# 1 Preliminary topics

## Objectives

- To revise the properties of **sine**, **cosine** and **tangent**.
- To revise the **sine rule** and the **cosine rule**.
- To revise **arithmetic sequences and series**.
- To revise **geometric sequences and series**.
- To revise **infinite geometric series**.
- To revise sequences defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r$ and $d$ are constants.
- To revise the **modulus function**.
- To sketch graphs of **circles**, **ellipses** and **hyperbolas** from their Cartesian equations.
- To revise the use of **parametric equations** to describe curves in the plane.
- To revise the use of **pseudocode** to describe algorithms.

In this chapter, we revise some of the knowledge and skills from Specialist Mathematics Units 1 & 2 that will be required in this course. We start by revising basic trigonometry, including the sine and cosine rules. There is further revision of trigonometry in Chapter 3.

We also revise sequences and series, the modulus function and the description of circles, ellipses and hyperbolas in the plane by Cartesian equations and by parametric equations. Finally, we revise the use of pseudocode to describe algorithms. For further details on these topics, refer to the relevant chapters of Specialist Mathematics Units 1 & 2.

We will be building on the introduction to parametric equations given in Section 1G in several new contexts in later chapters of this book.

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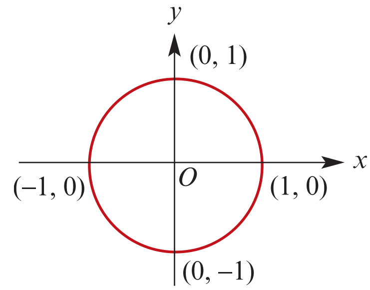
# 1A Circular functions

## Defining sine, cosine and tangent

The unit circle is a circle of radius 1 with centre at the origin. It is the graph of the relation $x^2 + y^2 = 1$.

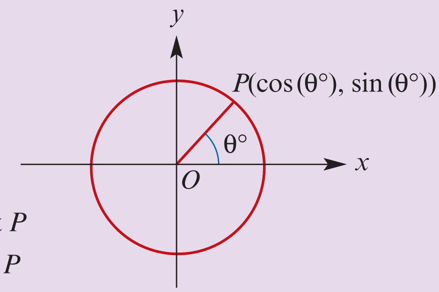



We can define the sine and cosine of any angle by using the unit circle.

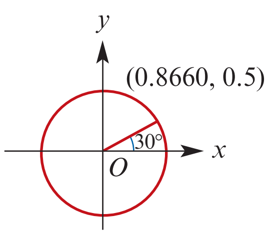
### Definition of sine and cosine

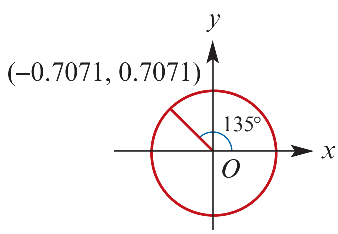
For each angle $\theta°$, there is a point $P$ on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the $x$-axis.

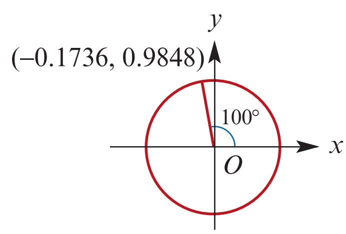
- $\cos(\theta°)$ is defined as the $x$-coordinate of the point $P$
- $\sin(\theta°)$ is defined as the $y$-coordinate of the point $P$




For example:




$\sin 30° = 0.5$ (exact value)

$\cos 30° = \dfrac{\sqrt{3}}{2} \approx 0.8660$

$\sin 135° = \dfrac{1}{\sqrt{2}} \approx 0.7071$

$\cos 135° = \dfrac{-1}{\sqrt{2}} \approx -0.7071$

$\sin 100° \approx 0.9848$

$\cos 100° \approx -0.1736$

### Definition of tangent

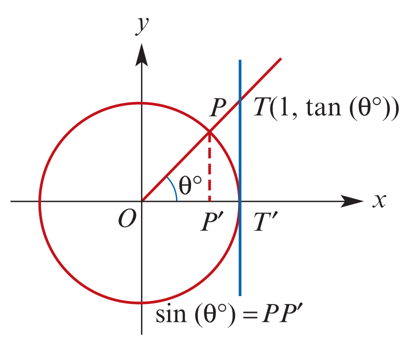
$$\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$$

The value of $\tan(\theta°)$ can be illustrated geometrically through the unit circle.

By considering similar triangles $OPP'$ and $OTT'$, it can be seen that

$$\frac{TT'}{OT'} = \frac{PP'}{OP'}$$

i.e. $$TT' = \frac{\sin(\theta°)}{\cos(\theta°)} = \tan(\theta°)$$

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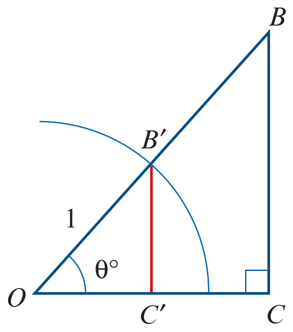
## The trigonometric ratios




For a right-angled triangle $OBC$, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

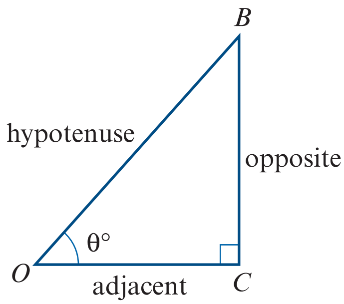
$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$

As triangles $OBC$ and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

$$\therefore \quad \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.



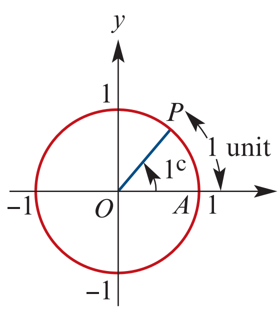

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$

## Definition of a radian

In moving around the unit circle a distance of 1 unit from $A$ to $P$, the angle $POA$ is defined. The measure of this angle is 1 radian.




One **radian** (written $1^c$) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

## Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Usually the symbol for radians, $^c$, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

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The following table displays the conversions of some special angles from degrees to radians.

| Angle in degrees | 0° | 30° | 45° | 60° | 90° | 180° | 360° |
|---|---|---|---|---|---|---|---|
| Angle in radians | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $2\pi$ |

Some values for the trigonometric functions are given in the following table.

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | undef |

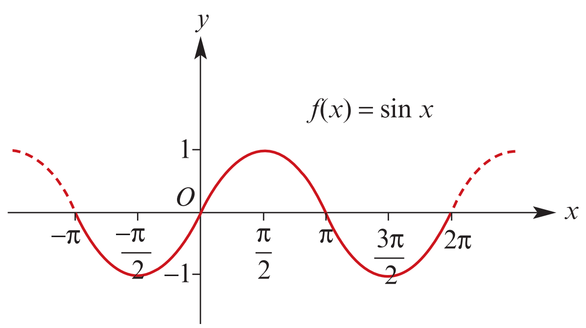
## The graphs of sine and cosine

A sketch of the graph of

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$$

is shown opposite.

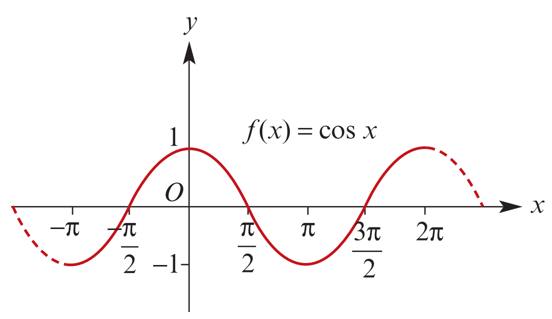
As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**. The period is $2\pi$. The amplitude is 1.




A sketch of the graph of

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$$

is shown opposite. The period of the cosine function is $2\pi$. The amplitude is 1.




For the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, where $a > 0$ and $n > 0$:

- Period $= \dfrac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

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## Symmetry properties of sine and cosine

The following results may be obtained from the graphs of the functions or from the unit-circle definitions:

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta \qquad \cos(\pi + \theta) = -\cos\theta$$
$$\sin(2\pi - \theta) = -\sin\theta \qquad \cos(2\pi - \theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\sin(\theta + 2n\pi) = \sin\theta \qquad \cos(\theta + 2n\pi) = \cos\theta \qquad \text{for } n \in \mathbb{Z}$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

### Example 1

**a** Convert 135° to radians. **b** Convert $1.5^c$ to degrees, correct to two decimal places.

**Solution**

**a** $135^\circ = \dfrac{135 \times \pi^c}{180} = \dfrac{3\pi^c}{4}$

**b** $1.5^c = \dfrac{1.5 \times 180^\circ}{\pi} = 85.94^\circ$ to two decimal places

### Example 2

Find the exact value of:

**a** $\sin 150^\circ$

**b** $\cos(-585^\circ)$

**Solution**

**a**
$$\begin{aligned}\sin 150^\circ &= \sin(180^\circ - 150^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

**b**
$$\begin{aligned}\cos(-585^\circ) &= \cos 585^\circ\\ &= \cos(585^\circ - 360^\circ)\\ &= \cos 225^\circ\\ &= -\cos 45^\circ\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

### Example 3

Find the exact value of:

**a** $\sin\left(\dfrac{11\pi}{6}\right)$

**b** $\cos\left(-\dfrac{45\pi}{6}\right)$

**Solution**

**a**
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

**b**
$$\begin{aligned}\cos\left(-\frac{45\pi}{6}\right) &= \cos(-7\tfrac{1}{2} \times \pi)\\ &= \cos\left(\frac{\pi}{2}\right)\\ &= 0\end{aligned}$$

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## The Pythagorean identity

For any value of θ:

$$\cos^2 \theta + \sin^2 \theta = 1$$

### Example 4

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, find:

**a** $\cos x$

**b** $\tan x$

**Solution**

**a** $\cos^2 x + \sin^2 x = 1$

$\cos^2 x + 0.09 = 1$

$\cos^2 x = 0.91$

$\therefore \quad \cos x = \pm\sqrt{0.91}$

Since $0 < x < \frac{\pi}{2}$, this gives

$$\cos x = \sqrt{0.91} = \sqrt{\frac{91}{100}} = \frac{\sqrt{91}}{10}$$

**b** $\tan x = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}}$

$= \frac{3}{\sqrt{91}}$

$= \frac{3\sqrt{91}}{91}$

## Solution of equations involving sine and cosine

If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

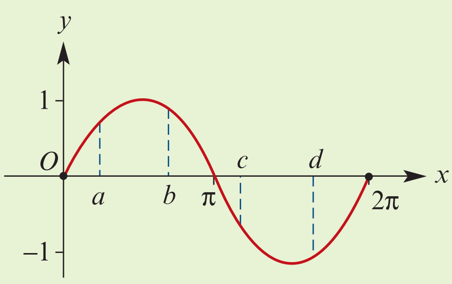
### Example 5

The graph of $y = f(x)$ for

$$f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \sin x$$

is shown.

For each pronumeral marked on the $x$-axis, find the other $x$-value which has the same $y$-value.




**Solution**

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

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## Example 6

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

### Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \le x \le 2\pi \quad &\Leftrightarrow \quad 0 \le 2x \le 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \ldots$$

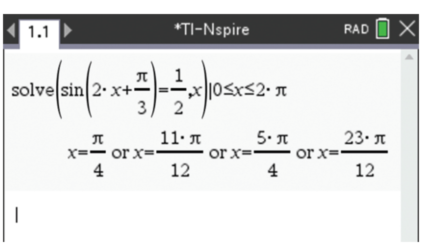
The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for $\theta$.

For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\begin{aligned} \theta &= \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \\ \therefore \quad 2x + \frac{2\pi}{6} &= \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \\ \therefore \quad 2x &= \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6} \\ \therefore \quad x &= \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12} \end{aligned}$$

### Using the TI-Nspire

- Ensure your calculator is in radian mode. (To change the angle mode, either go to (⌂ on) > **Settings** > **Document Settings** or else hover the cursor over **RAD** or **DEG** at the top of the screen and click to toggle modes.)
- Complete as shown.




Note: The **Graph** application has its own settings, which are accessed from a **Graph** page using (menu) > **Settings**.

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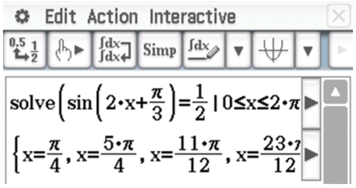
### Using the Casio ClassPad

- Open the Main $\sqrt{\alpha}$ application.
- Ensure your calculator is in radian mode (with **Rad** in the status bar at the bottom of the main screen).
- Enter and highlight

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \mid 0 \le x \le 2\pi$$

- Select **Interactive** > **Equation/Inequality** > **solve**.
- Tap ► on the solution line to view the entire solution.




## Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a\sin(nx + \varepsilon) + b \qquad \text{and} \qquad f(x) = a\cos(nx + \varepsilon) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

### Example 7

Sketch the graph of the function

$$h\colon [0, 2\pi] \to \mathbb{R},\ h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1$$

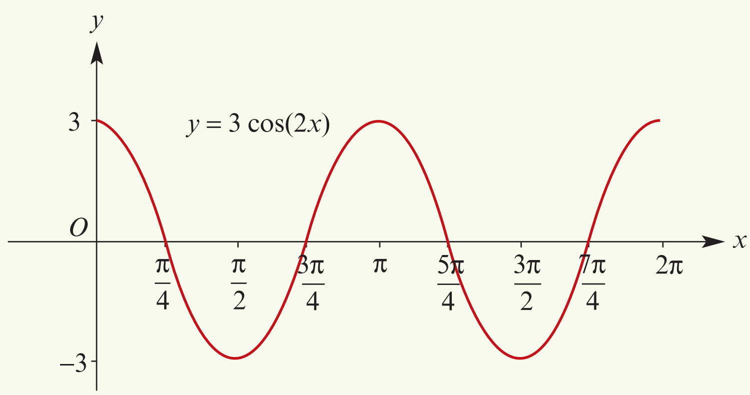
#### Solution

We can write $h(x) = 3\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$.

The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ from the $y$-axis
- a dilation of factor 3 from the $x$-axis
- a translation of $\frac{\pi}{6}$ units in the negative direction of the $x$-axis
- a translation of 1 unit in the positive direction of the $y$-axis.

First apply the two dilations to the graph of $y = \cos x$.

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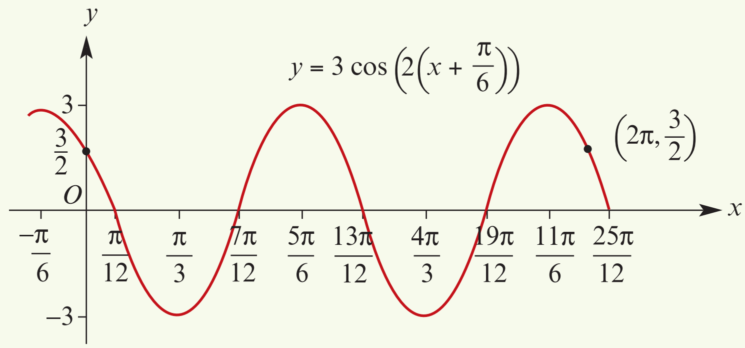
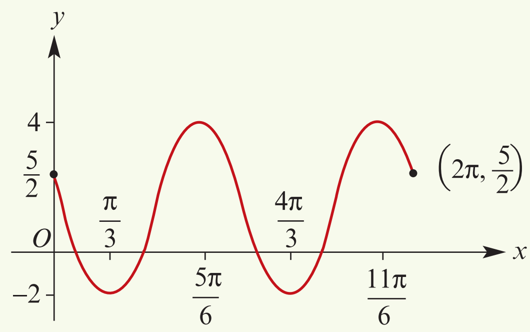
Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the $x$-axis.

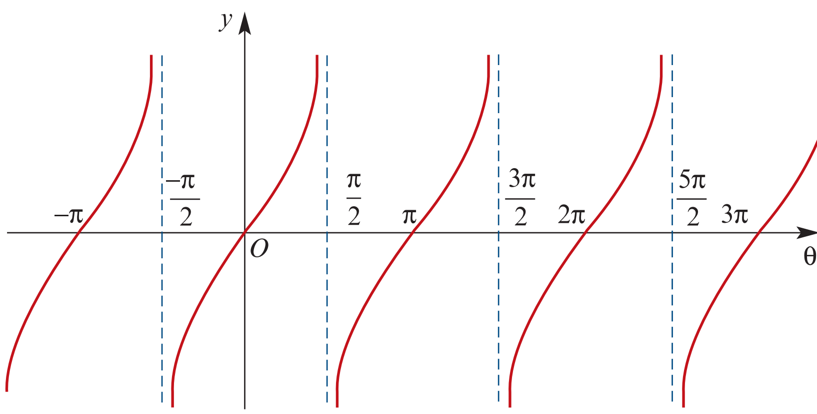



Apply the final translation and restrict the graph to the required domain.




## The graph of tan

A sketch of the graph of $y = \tan\theta$ is shown below.




Notes:

- The domain of tan is $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$.
- The range of tan is $\mathbb{R}$.
- The graph repeats itself every $\pi$ units, i.e. the period of tan is $\pi$.
- The vertical asymptotes have equations $\theta = \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$.

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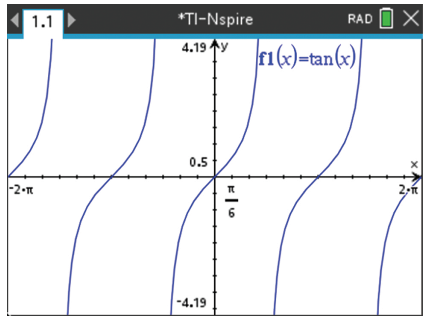
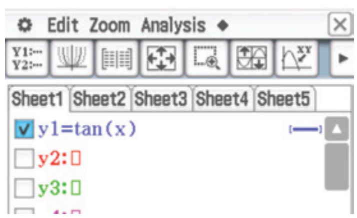
### Using the TI-Nspire

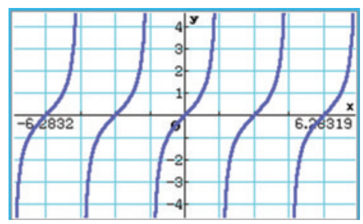
- Open a **Graphs** application and define $f1(x) = \tan(x)$.
- Press (enter) to obtain the graph.
- To change the viewing window, go to (menu) > **Window/Zoom** > **Window Settings**.




### Using the Casio ClassPad

- Open the menu Menu; select **Graph & Table** Graph& Table.
- Enter $\tan(x)$ in $y1$, tick the box and tap .
- If necessary, select **Zoom** > **Quick** > **Quick Trig** or tap  to manually adjust the window. In the graph shown below, the $x$-axis scale has been set to $\frac{\pi}{2}$.




## Symmetry properties of tan

The following results are obtained from the definition of tan:

$$\tan(\pi - \theta) = -\tan\theta \qquad \tan(2\pi - \theta) = -\tan\theta$$

$$\tan(\pi + \theta) = \tan\theta \qquad \tan(-\theta) = -\tan\theta$$

### Example 8

Find the exact value of:

**a** $\tan 330°$

**b** $\tan\left(\dfrac{4\pi}{3}\right)$

#### Solution

**a** $\tan 330° = \tan(360° - 30°)$

$= -\tan 30°$

$= -\dfrac{1}{\sqrt{3}}$

**b** $\tan\left(\dfrac{4\pi}{3}\right) = \tan\left(\pi + \dfrac{\pi}{3}\right)$

$= \tan\left(\dfrac{\pi}{3}\right)$

$= \sqrt{3}$

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## Solution of equations involving tan

The procedure here is similar to that used for solving equations involving sin and cos, except that only one solution needs to be selected then all other solutions are one period length apart.

### Example 9

Solve the following equations:

**a** $\tan x = -1$ for $x \in [0, 4\pi]$

**b** $\tan(2x - \pi) = \sqrt{3}$ for $x \in [-\pi, \pi]$

**Solution**

**a** $\tan x = -1$

Now $\tan\left(\frac{3\pi}{4}\right) = -1$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{3\pi}{4} + \pi \text{ or } \frac{3\pi}{4} + 2\pi \text{ or } \frac{3\pi}{4} + 3\pi$$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{11\pi}{4} \text{ or } \frac{15\pi}{4}$$

**b** Let $\theta = 2x - \pi$. Then

$$\begin{aligned} -\pi \le x \le \pi &\Leftrightarrow -2\pi \le 2x \le 2\pi \\ &\Leftrightarrow -3\pi \le 2x - \pi \le \pi \\ &\Leftrightarrow -3\pi \le \theta \le \pi \end{aligned}$$

To solve $\tan(2x - \pi) = \sqrt{3}$, we first solve $\tan\theta = \sqrt{3}$.

$$\theta = \frac{\pi}{3} \text{ or } \frac{\pi}{3} - \pi \text{ or } \frac{\pi}{3} - 2\pi \text{ or } \frac{\pi}{3} - 3\pi$$

$$\therefore \quad \theta = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{2} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x - \pi = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x = \frac{4\pi}{3} \text{ or } \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3}$$

$$\therefore \quad x = \frac{2\pi}{3} \text{ or } \frac{\pi}{6} \text{ or } -\frac{\pi}{3} \text{ or } -\frac{5\pi}{6}$$

Skill-sheet

## Exercise 1A

Example 1

**1** **a** Convert the following angles from degrees to exact values in radians:

**i** 720° **ii** 540° **iii** −450° **iv** 15° **v** −10° **vi** −315°

**b** Convert the following angles from radians to degrees:

**i** $\frac{5\pi}{4}$ **ii** $-\frac{2\pi}{3}$ **iii** $\frac{7\pi}{12}$ **iv** $-\frac{11\pi}{6}$ **v** $\frac{13\pi}{9}$ **vi** $-\frac{11\pi}{12}$

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**2** Perform the correct conversion on each of the following angles, giving the answer correct to two decimal places.

**a** Convert from degrees to radians:

**i** $7°$ **ii** $-100°$ **iii** $-25°$ **iv** $51°$ **v** $206°$ **vi** $-410°$

**b** Convert from radians to degrees:

**i** $1.7^c$ **ii** $-0.87^c$ **iii** $2.8^c$ **iv** $0.1^c$ **v** $-3^c$ **vi** $-8.9^c$

Example 2

**3** Find the exact value of each of the following:

**a** $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$

**d** $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

Example 3

**4** Find the exact value of each of the following:

**a** $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\cos\left(-\frac{\pi}{3}\right)$

**d** $\cos\left(\frac{5\pi}{4}\right)$ **e** $\cos\left(\frac{9\pi}{4}\right)$ **f** $\sin\left(\frac{11\pi}{3}\right)$

**g** $\cos\left(\frac{31\pi}{6}\right)$ **h** $\cos\left(\frac{29\pi}{6}\right)$ **i** $\sin\left(-\frac{23\pi}{6}\right)$

Example 4

**5** If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, find:

**a** $\cos x$ **b** $\tan x$

**6** If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, find:

**a** $\sin x$ **b** $\tan x$

**7** If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find:

**a** $\cos x$ **b** $\tan x$

**8** If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find:

**a** $\cos x$ **b** $\tan x$

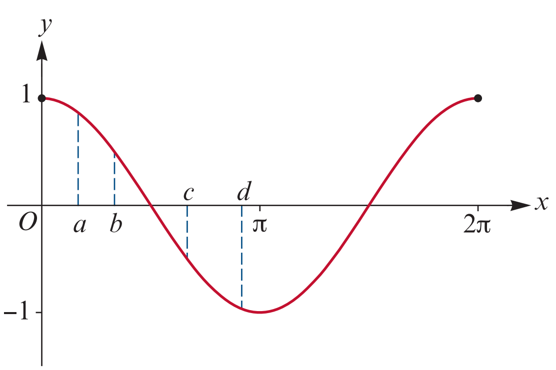
Example 5

**9** The graph of $y = f(x)$ for

$f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \cos x$

is shown.

For each pronumeral marked on the $x$-axis, find the other $x$-value which has the same $y$-value.

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Example 6 **10** Solve each of the following for $x \in [0, 2\pi]$:

**a** $\sin x = -\frac{\sqrt{3}}{2}$ **b** $\sin(2x) = -\frac{\sqrt{3}}{2}$

**c** $2\cos(2x) = -1$ **d** $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$

**e** $2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \frac{\pi}{3}\right) = -\sqrt{3}$

Example 7 **11** Sketch the graph of each of the following for the stated domain:

**a** $f(x) = \sin(2x),\ x \in [0, 2\pi]$ **b** $f(x) = \cos\left(x + \frac{\pi}{3}\right),\ x \in \left[-\frac{\pi}{3}, \pi\right]$

**c** $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right),\ x \in [0, \pi]$ **d** $f(x) = 2\sin(3x) + 1,\ x \in [0, \pi]$

**e** $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3},\ x \in [0, 2\pi]$

Example 8 **12** Find the exact value of each of the following:

**a** $\tan\left(\frac{5\pi}{4}\right)$ **b** $\tan\left(-\frac{2\pi}{3}\right)$ **c** $\tan\left(-\frac{29\pi}{6}\right)$ **d** $\tan 240°$

**13** If $\tan x = \frac{1}{4}$ and $\pi \le x \le \frac{3\pi}{2}$, find the exact value of:

**a** $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

**14** If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find the exact value of:

**a** $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 9 **15** Solve each of the following for $x \in [0, 2\pi]$:

**a** $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

**c** $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

**16** Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

**a** $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

**c** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

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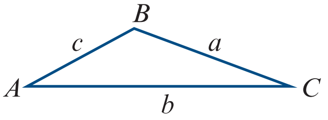
# 1B The sine and cosine rules

In this section, we revise methods for finding unknown quantities (side lengths or angles) in a non-right-angled triangle.

### Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.




For example, the magnitude of angle $BAC$ is denoted by $A$, and the length of side $BC$ is denoted by $a$.

## The sine rule

The sine rule is used to find unknown quantities in a triangle in the following two situations:

**1** one side and two angles are given

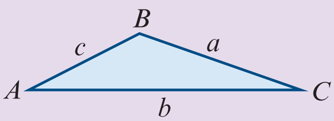
**2** two sides and a non-included angle are given.

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

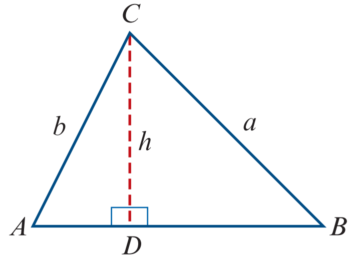
**Sine rule**

For triangle $ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$




**Proof** We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.




In triangle $ACD$:

$$\sin A = \frac{h}{b}$$

$$\therefore \quad h = b \sin A$$

In triangle $BCD$:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from $A$ to $BC$ would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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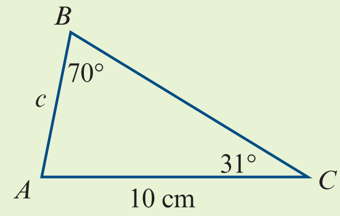
## Example 10

Use the sine rule to find the length of $AB$.




### Solution

$$\frac{c}{\sin 31°} = \frac{10}{\sin 70°}$$

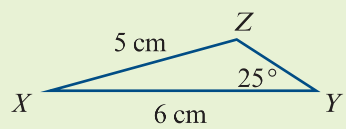
$$\therefore \quad c = \frac{10 \sin 31°}{\sin 70°}$$

$$= 5.4809\ldots$$

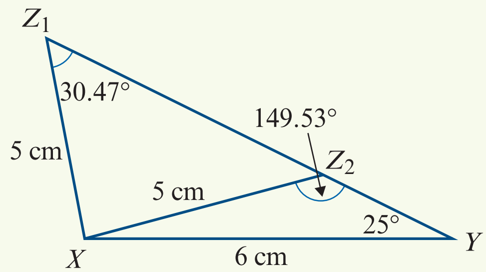
The length of $AB$ is 5.48 cm, correct to two decimal places.

## Example 11

Use the sine rule to find the magnitude of angle $XZY$, given that $Y = 25°$, $y = 5$ and $z = 6$.




### Solution




$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5}$$

$$= 0.5071\ldots$$

$$\therefore \quad Z = (30.473\ldots)° \quad \text{or} \quad Z = (180 - 30.473\ldots)°$$

Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

**Notes:**

- Remember that $\sin(180 - \theta)° = \sin(\theta°)$.
- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles. An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

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## The cosine rule

The cosine rule can be used to find unknown quantities in a triangle in the following two situations:

1. two sides and the included angle are given
2. three sides are given.

In each case, the triangle is uniquely defined up to congruence.

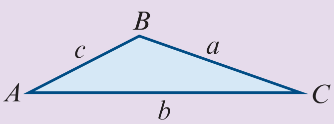
### Cosine rule

For triangle $ABC$:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$




The symmetrical results also hold:

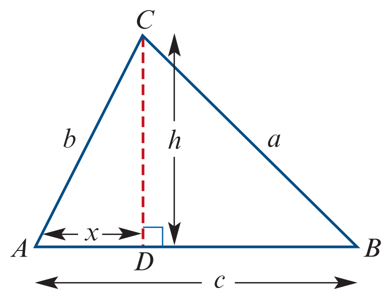
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

**Proof** We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle $ACD$:

$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b\cos A$$




Using Pythagoras' theorem in triangles $ACD$ and $BCD$:

$$b^2 = x^2 + h^2$$

$$a^2 = (c - x)^2 + h^2$$

Expanding gives

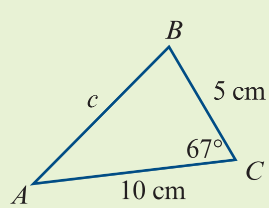
$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

### Example 12

For triangle $ABC$, find the length of $AB$ in centimetres correct to two decimal places.

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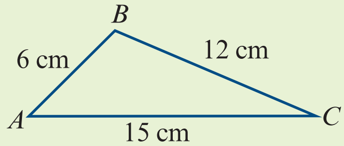
**Solution**

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

The length of $AB$ is 9.27 cm, correct to two decimal places.

## Example 13

For triangle $ABC$, find the magnitude of angle $ABC$ correct to two decimal places.




**Solution**

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

The magnitude of angle $ABC$ is 108.21°, correct to two decimal places.

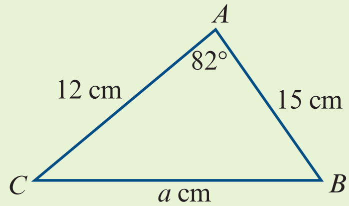
## Example 14

In $\triangle ABC$, $\angle CAB = 82^\circ$, $AC = 12$ cm and $AB = 15$ cm.

Find correct to two decimal places:

**a** $BC$

**b** $\angle ACB$




**Solution**

**a** Find $BC$ using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$= 12^2 + 15^2 - 2 \times 12 \times 15 \cos 82^\circ$$
$$= 144 + 225 - 360 \cos 82^\circ$$
$$= 318.8976\ldots$$
$$a = 17.8577\ldots$$

Thus $BC = a = 17.86$ cm, correct to two decimal places.

**b** Find $\angle ACB$ using the sine rule:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\therefore \quad \sin C = \frac{c \sin A}{a}$$
$$= \frac{15 \sin 82^\circ}{17.8577}$$

Thus $\angle ACB = 56.28^\circ$, correct to two decimal places.

**Note:** In part **b**, the angle $C = 123.72^\circ$ is also a solution to the equation, but it is discarded as a possible answer because it is inconsistent with the given angle $A = 82^\circ$.

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Skill-sheet

## Exercise 1B

Example 10

**1** In triangle $ABC$, $\angle BAC = 73^\circ$, $\angle ACB = 55^\circ$ and $AB = 10$ cm. Find correct to two decimal places:

**a** $BC$ **b** $AC$

Example 11

**2** In $\triangle ABC$, $\angle ACB = 34^\circ$, $AC = 8.5$ cm and $AB = 5.6$ cm. Find correct to two decimal places:

**a** the two possible values of $\angle ABC$ (one acute and one obtuse)

**b** $BC$ in each case.

Example 12

**3** In triangle $ABC$, $\angle ABC = 58^\circ$, $AB = 6.5$ cm and $BC = 8$ cm. Find correct to two decimal places:

**a** $AC$ **b** $\angle BCA$

Example 13
Example 14

**4** In $\triangle ABC$, $AB = 5$ cm, $BC = 12$ cm and $AC = 10$ cm. Find:

**a** the magnitude of $\angle ABC$, correct to two decimal places

**b** the magnitude of $\angle BAC$, correct to two decimal places.

**5** The adjacent sides of a parallelogram are 9 cm and 11 cm. One of its angles is 67°. Find the length of the longer diagonal, correct to two decimal places.

Example 14

**6** In $\triangle ABC$, $\angle ABC = 35^\circ$, $AB = 10$ cm and $BC = 4.7$ cm. Find correct to two decimal places:

**a** $AC$ **b** $\angle ACB$

**7** In $\triangle ABC$, $\angle ABC = 45^\circ$, $\angle ACB = 60^\circ$ and $AC = 12$ cm. Find $AB$.

**8** In $\triangle PQR$, $\angle QPR = 60^\circ$, $PQ = 2$ cm and $PR = 3$ cm. Find $QR$.

**9** In $\triangle ABC$, the angle $ABC$ has magnitude $40^\circ$, $AC = 20$ cm and $AB = 18$ cm. Find the distance $BC$ correct to two decimal places.

**10** In $\triangle ABC$, the angle $ACB$ has magnitude $30^\circ$, $AC = 10$ cm and $AB = 8$ cm. Find the distance $BC$ using the cosine rule.

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# 1C Sequences and series

The following are examples of sequences of numbers:

**a** $1, 3, 5, 7, 9, \ldots$ **b** $10, 7, 4, 1, -2, \ldots$ **c** $0.6, 1.7, 2.8, \ldots, 9.4$

**d** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ **e** $0.1, 0.11, 0.111, 0.1111, 0.11111, \ldots$

Each sequence is a list of numbers, with order being important.

The numbers of a sequence are called its **terms**. The $n$th term of a sequence is denoted by the symbol $t_n$. So the first term is $t_1$, the 12th term is $t_{12}$, and so on.

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation**, a **recursive formula** or an **iterative rule**. For example:

- The sequence $1, 3, 5, 7, 9, \ldots$ may be defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ may be defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

## Example 15

Use the recurrence relation to find the first four terms of the sequence

$$t_1 = 3, \quad t_n = t_{n-1} + 5$$

### Solution

$$t_1 = 3$$
$$t_2 = t_1 + 5 = 8$$
$$t_3 = t_2 + 5 = 13$$
$$t_4 = t_3 + 5 = 18$$

The first four terms are $3, 8, 13, 18$.

## Example 16

Find a possible recurrence relation for the following sequence:

$$9, -3, 1, -\frac{1}{3}, \ldots$$

### Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

The sequence is defined by $t_1 = 9$ and $t_n = -\frac{1}{3}t_{n-1}$.

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A sequence may also be defined explicitly by a rule that is stated in terms of $n$. For example:

- The rule $t_n = 2n$ defines the sequence $t_1 = 2,\ t_2 = 4,\ t_3 = 6,\ t_4 = 8, \dots$
- The rule $t_n = 2^{n-1}$ defines the sequence $t_1 = 1,\ t_2 = 2,\ t_3 = 4,\ t_4 = 8, \dots$
- The sequence $1, 3, 5, 7, 9, \dots$ can be defined by $t_n = 2n - 1$.
- The sequence $t_1 = \frac{1}{3},\ t_n = \frac{1}{3}t_{n-1}$ can be defined by $t_n = \frac{1}{3^n}$.

**Example 17**

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

**Solution**

$$t_1 = 2(1) + 3 = 5$$
$$t_2 = 2(2) + 3 = 7$$
$$t_3 = 2(3) + 3 = 9$$
$$t_4 = 2(4) + 3 = 11$$

The first four terms are $5, 7, 9, 11$.

## Arithmetic sequences

A sequence in which each successive term is found by adding a fixed amount to the previous term is called an **arithmetic sequence**. That is, an arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where $d$ is a constant.

For example: $2, 5, 8, 11, 14, 17, \dots$ is an arithmetic sequence.

The $n$th term of an arithmetic sequence is given by

$$t_n = a + (n-1)d$$

where $a$ is the first term and $d$ is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

**Example 18**

Find the 10th term of the arithmetic sequence $-4, -1, 2, 5, \dots$.

**Solution**

$a = -4,\ d = 3$

$$t_n = a + (n-1)d$$
$$\therefore\quad t_{10} = -4 + (10-1) \times 3 = 23$$

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## Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence is arithmetic, then the series is called an **arithmetic series**.

The symbol $S_n$ is used to denote the sum of the first $n$ terms of a sequence. That is,

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

Writing this sum in reverse order, we have

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a$$

Adding these two expressions together gives

$$2S_n = n(2a + (n - 1)d)$$

Therefore

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

Since the last term $\ell = t_n = a + (n - 1)d$, we can also write

$$S_n = \frac{n}{2}(a + \ell)$$

## Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. That is, a geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where $r$ is a constant.

For example: $2, 6, 18, 54, \ldots$ is a geometric sequence.

The $n$th term of a geometric sequence is given by

$$t_n = ar^{n-1}$$

where $a$ is the first term and $r$ is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

### Example 19

Find the 10th term of the geometric sequence $2, 6, 18, \ldots$.

**Solution**

$a = 2,\ r = 3$

$$\begin{aligned} t_n &= ar^{n-1} \\ \therefore \quad t_{10} &= 2 \times 3^{10-1} \\ &= 39\,366 \end{aligned}$$

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## Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for $S_n$, the sum of the first $n$ terms of a geometric sequence, can be found using a similar method to that used for arithmetic series.

Let $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (1)$

Then $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad (2)$

Subtract (1) from (2):

$$rS_n - S_n = ar^n - a$$
$$S_n(r-1) = a(r^n - 1)$$

Therefore

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

For values of $r$ such that $-1 < r < 1$, it is often more convenient to use the equivalent formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by multiplying both the numerator and the denominator by $-1$.

**Example 20**

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$.

**Solution**

$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n = 9$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(1 - (\frac{1}{3})^9\right)}{1 - \frac{1}{3}}$$
$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^9\right)$$
$$\approx 0.499975$$

## Infinite geometric series

If a geometric sequence has a common ratio with magnitude less than 1, that is, if $-1 < r < 1$, then each successive term is closer to zero. For example, consider the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

In Example 20 we found that the sum of the first 9 terms is $S_9 \approx 0.499975$. The sum of the first 20 terms is $S_{20} \approx 0.49999999986$. We might conjecture that, as we add more and more terms of the sequence, the sum will get closer and closer to 0.5, that is, $S_n \to 0.5$ as $n \to \infty$.

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An infinite series $t_1 + t_2 + t_3 + \cdots$ is said to be **convergent** if the sum of the first $n$ terms, $S_n$, approaches a limiting value as $n \to \infty$. This limit is called the **sum to infinity** of the series.

If $-1 < r < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots$ is convergent and the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

**Proof** We know that

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \to \infty$, we have $r^n \to 0$ and so $\frac{ar^n}{1-r} \to 0$. Hence $S_n \to \frac{a}{1-r}$ as $n \to \infty$.

**Example 21**

Find the sum to infinity of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

**Solution**

$a = \frac{1}{2}$, $r = \frac{1}{2}$ and therefore

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

## Using a CAS calculator with sequences

**Example 22**

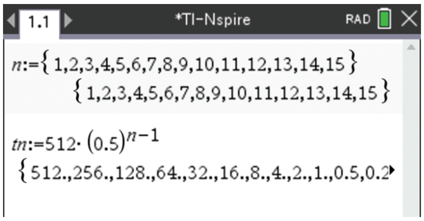
Use a calculator to generate terms of the geometric sequence defined by

$$t_n = 512(0.5)^{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

### Using the TI-Nspire

Sequences defined in terms of $n$ can be investigated in a **Calculator** application.

- To generate the first 15 terms of the sequence defined by the rule $t_n = 512(0.5)^{n-1}$, complete as shown.




Note: Alternatively, assign these values to $n$ by entering $n :=$ seq($k$, $k$, 1, 15, 1).
Assigning (storing) the resulting list as $tn$ enables the sequence to be graphed.
The lists $n$ and $tn$ can also be created in a **Lists & Spreadsheet** application.

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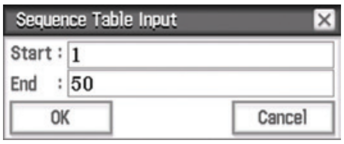
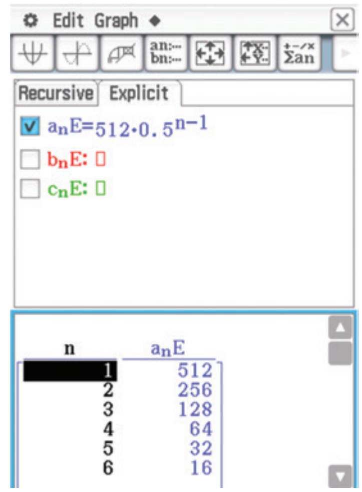
## Using the Casio ClassPad

- Open the menu Menu; select **Sequence** Sequence.
- Ensure that the **Explicit** window is activated.
- Tap the cursor next to $a_nE$ and enter $512 \times 0.5^{n-1}$. (The variable $n$ can be entered by tapping on n in the toolbar.)
- Tick the box or tap EXE.
- Tap to view the sequence values.
- Tap to open the Sequence Table Input window and complete as shown below; tap OK.



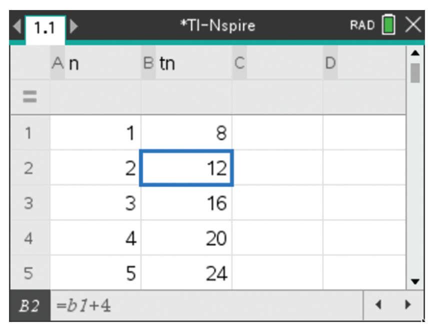



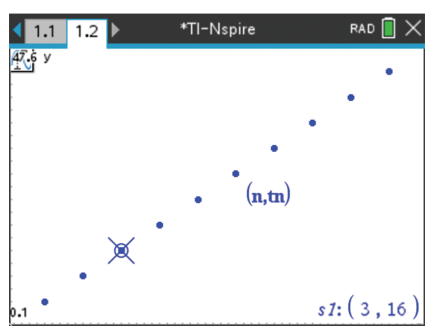

## Example 23

Use a CAS calculator to plot the graph of the arithmetic sequence defined by the recurrence relation $t_n = t_{n-1} + 4$ and $t_1 = 8$.

## Using the TI-Nspire

- In a **Lists & Spreadsheet** page, name the first two columns $n$ and $tn$ respectively.
- Enter 1 in cell A1 and enter 8 in cell B1.
- Enter $= a1 + 1$ in cell A2 and enter $= b1 + 4$ in cell B2.
- Highlight the cells A2 and B2 using shift and the arrows.
- Use menu > **Data** > **Fill** to fill down to row 10 and press enter. This generates the first 10 terms of the sequence.
- To graph the sequence, open a **Graphs** application (ctrl I > **Add Graphs**).
- Create a scatter plot using menu > **Graph Entry/Edit** > **Scatter Plot**. Enter the list variables as $n$ and $tn$ in their respective fields.
- Set an appropriate window using menu > **Window/Zoom** > **Zoom – Data**.




Note: It is possible to see the coordinates of the points: menu > **Trace** > **Graph Trace**. The scatter plot can also be graphed in a **Data & Statistics** page.

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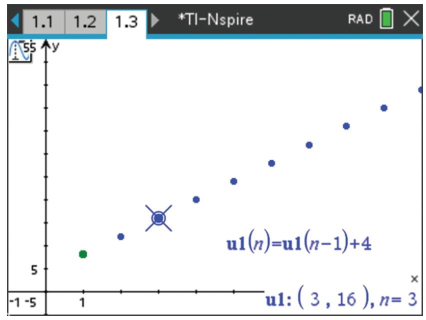
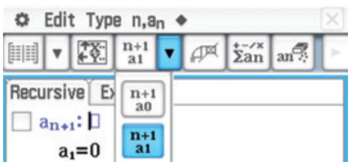
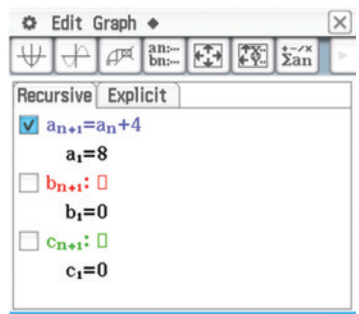
- Alternatively, the sequence can be graphed directly in the sequence plotter (menu > **Graph Entry/Edit** > **Sequence** > **Sequence**) with initial value 8.




## Using the Casio ClassPad

- Open the menu Menu; select **Sequence**.
- Ensure that the **Recursive** window is activated.
- Select the setting $\frac{n+1}{a1}$ as shown below.

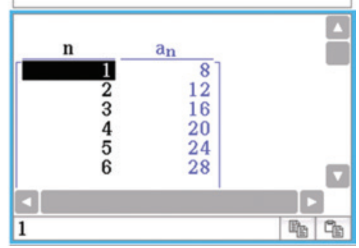



- Tap the cursor next to $a_{n+1}$ and enter $a_n + 4$.

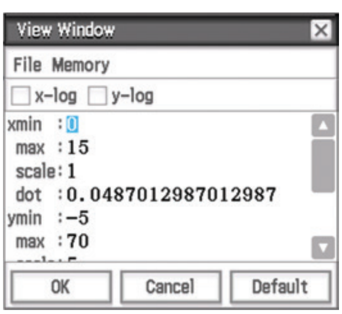
Note: The symbol $a_n$ can be found in the dropdown menu $n, a_n$.

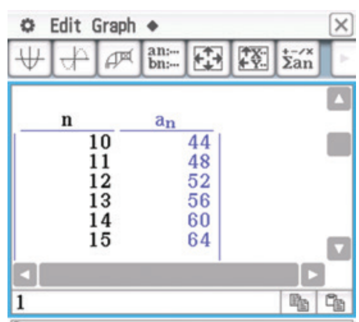
- Enter 8 for the value of the first term, $a_1$.
- Tick the box. Tap [icon] to view the sequence values.



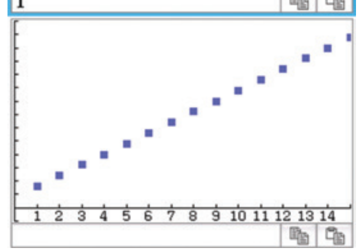
| n | an |
|---|---|
| 1 | 8 |
| 2 | 12 |
| 3 | 16 |
| 4 | 20 |
| 5 | 24 |
| 6 | 28 |

- Tap [icon] to view the graph.
- Tap [icon] and adjust the window setting for the first 15 terms as shown below.



| n | an |
|---|---|
| 10 | 44 |
| 11 | 48 |
| 12 | 52 |
| 13 | 56 |
| 14 | 60 |
| 15 | 64 |



- Select **Analysis** > **Trace** and use the cursor ► to view each value in the sequence.

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## Recurrence relations of the form $t_n = rt_{n-1} + d$

We now consider a generalisation of arithmetic and geometric sequences. We shall study sequences defined by a recurrence relation of the form

$$t_n = rt_{n-1} + d$$

where $r$ and $d$ are constants.

Note: The case where $r = 1$ corresponds to an arithmetic sequence.
The case where $d = 0$ corresponds to a geometric sequence.

We can establish a general formula for the $n$th term of the sequence.

For a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r \neq 1$, the $n$th term is given by

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

where $t_1$ is the first term.

**Proof** We can establish the formula by checking that it gives the correct first term and that it satisfies the recurrence relation.

First term Using the formula to find the first term ($n = 1$) gives

$$r^{1-1}t_1 + \frac{d(r^{1-1} - 1)}{r - 1} = 1 \times t_1 + 0 = t_1$$

which is correct.

Recurrence relation We now check that the formula satisfies the recurrence relation $t_n = rt_{n-1} + d$. Starting from the right-hand side:

$$\begin{aligned} rt_{n-1} + d &= r\left(r^{n-2}t_1 + \frac{d(r^{n-2} - 1)}{r - 1}\right) + d && \text{(using the formula for } t_{n-1}\text{)} \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - r)}{r - 1} + d \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - r)}{r - 1} + \frac{d(r - 1)}{r - 1} \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \\ &= t_n && \text{(using the formula for } t_n\text{)} \end{aligned}$$

So the recurrence relation holds.

We have shown that the formula gives the correct value for $t_1$. Since it satisfies the recurrence relation, this means that $t_2$ is correct, and then this means that $t_3$ is correct, and so on. (This proof uses mathematical induction, which is revised in Chapter 2.)

Note: This general formula for $t_n$ can be rewritten into a rule of the form $t_n = Ar^{n-1} + B$, for constants $A$ and $B$. We use this observation in Example 25.

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### Example 24

Find a formula for the $n$th term of the sequence defined by the recurrence relation

$$t_n = 2t_{n-1} + 1, \quad t_1 = 10$$

**Solution**

We will use the general formula

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

Here $r = 2$ and $d = 1$. Hence

$$\begin{aligned} t_n &= 2^{n-1} \times 10 + \frac{1 \times (2^{n-1} - 1)}{2 - 1} \\ &= 10 \times 2^{n-1} + 2^{n-1} - 1 \\ &= 11 \times 2^{n-1} - 1 \end{aligned}$$

### Example 25

The sequence 5, 16, 38, . . . is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the $n$th term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants $A$ and $B$.

**Solution**

From the first three terms, we have

$$t_1 = A + B = 5 \qquad (1)$$

$$t_2 = Ar + B = 16 \qquad (2)$$

$$t_3 = Ar^2 + B = 38 \qquad (3)$$

Subtract equation (1) from both (2) and (3):

$$A(r - 1) = 11 \qquad (4)$$

$$A(r^2 - 1) = 33 \qquad (5)$$

Divide (5) by (4):

$$\begin{aligned} \frac{r^2 - 1}{r - 1} &= 3 \\ \frac{(r + 1)(r - 1)}{r - 1} &= 3 \\ r + 1 &= 3 \\ r &= 2 \end{aligned}$$

Using (4) now gives $A = 11$, and using (1) gives $B = -6$.

The formula for the $n$th term is

$$t_n = 11 \times 2^{n-1} - 6$$

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## Exercise 1C

Example 15 **1** Use the recurrence relation to find the first four terms of the sequence $t_1 = 3$, $t_n = t_{n-1} - 4$.

Example 16 **2** Find a possible recurrence relation for the sequence $-2, 6, -18, \dots$.

Example 17 **3** Find the first four terms of the sequence defined by $t_n = 2n - 3$ for $n \in \mathbb{N}$.

**4** The Fibonacci sequence is given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_1 = F_2 = 1$. Find the first 10 terms of the Fibonacci sequence.

Example 18 **5** Find the 10th term of the arithmetic sequence $-4, -7, -10, \dots$.

Example 19 **6** Calculate the 10th term of the geometric sequence $2, -6, 18, \dots$.

**7** Find the sum of the first 10 terms of an arithmetic sequence with first term 3 and common difference 4.

Example 20 **8** Find the sum of the first eight terms of a geometric sequence with first term 6 and common ratio $-3$.

Example 21 **9** Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

**10** The first, second and third terms of a geometric sequence are $x + 5$, $x$ and $x - 4$ respectively. Find:

**a** the value of $x$

**b** the common ratio

**c** the difference between the sum to infinity and the sum of the first 10 terms.

**11** Find the sum to infinity of the geometric sequence $a, \frac{a}{\sqrt{2}}, \frac{a}{2}, \frac{a}{2\sqrt{2}}, \dots$ in terms of $a$.

**12** Consider the sum

$$S_n = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots + \frac{x^{n-1}}{2^{n-1}}$$

**a** Calculate $S_{10}$ when $x = 1.5$.

**b** **i** Find the possible values of $x$ for which the sum to infinity $S_\infty$ exists.

**ii** Find the values of $x$ for which $S_\infty = 2S_{10}$.

**13** **a** Find an expression for the sum to infinity of the infinite geometric series

$$1 + \sin\theta + \sin^2\theta + \cdots$$

**b** Find the values of $\theta$ for which the sum to infinity is 2.

Example 23 **14** A sequence is defined recursively by $t_1 = 6$, $t_{n+1} = 3t_n - 1$. Find $t_2$ and $t_3$. Use a CAS calculator to find $t_8$.

**15** A sequence is defined recursively by $y_1 = 5$, $y_{n+1} = 2y_n + 6$. Find $y_2$ and $y_3$. Use a CAS calculator to find $y_{10}$ and to plot a graph showing the first 10 terms.

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Example 24 **16** For each of the following recurrence relations, determine an expression for the $n$th term of the sequence in terms of $n$:

**a** $t_n = 2t_{n-1} - 8, \quad t_1 = 8$ **b** $t_n = 2t_{n-1} - 2, \quad t_1 = 10$ **c** $t_{n+1} = \frac{1}{2}t_n + 6, \quad t_1 = 40$

Example 25 **17** The sequence $6, 7, 9, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the $n$th term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants $A$ and $B$.

**18** The sequence $8, 23, 98, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the $n$th term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants $A$ and $B$.

# 1D The modulus function

The **modulus** or **absolute value** of a real number $x$ is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

## Example 26

Evaluate each of the following:

**a** **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

**b** **i** $\left|\frac{-4}{2}\right|$ **ii** $\frac{|-4|}{|2|}$

**c** **i** $|-6 + 2|$ **ii** $|-6| + |2|$

### Solution

**a** **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

**b** **i** $\left|\frac{-4}{2}\right| = |-2| = 2$ **ii** $\frac{|-4|}{|2|} = \frac{4}{2} = 2$ Note: $\left|\frac{-4}{2}\right| = \frac{|-4|}{|2|}$

**c** **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

### Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If $a$ and $b$ are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

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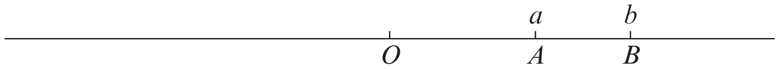
## The modulus function as a measure of distance

Consider two points $A$ and $B$ on a number line:




On a number line, the distance between points $A$ and $B$ is $|a - b| = |b - a|$.

Thus $|x - 2| \leq 3$ can be read as 'the distance of $x$ from 2 is less than or equal to 3', and $|x| \leq 3$ can be read as 'the distance of $x$ from the origin is less than or equal to 3'. Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

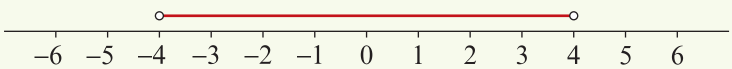
### Example 27

Illustrate each set on a number line and represent the set using interval notation:

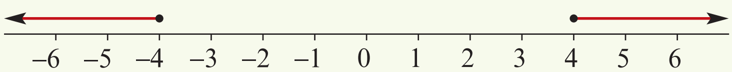
**a** $\{ x : |x| < 4 \}$ **b** $\{ x : |x| \geq 4 \}$ **c** $\{ x : |x - 1| \leq 4 \}$

#### Solution

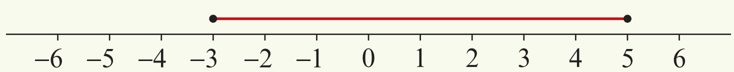
**a** $(-4, 4)$




**b** $(-\infty, -4] \cup [4, \infty)$



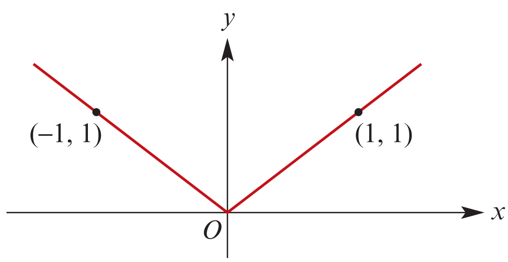

**c** $[-3, 5]$




## The graph of $y = |x|$

The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is shown here.

This graph is symmetric about the $y$-axis, since $|x| = |-x|$.




### Example 28

For each of the following functions, sketch the graph and state the range:

**a** $f(x) = |x - 3| + 1$ **b** $f(x) = -|x - 3| + 1$

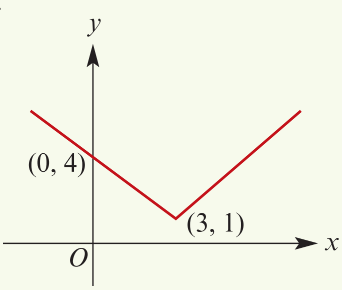
#### Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

**a** $$f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$$

Range $= [1, \infty)$

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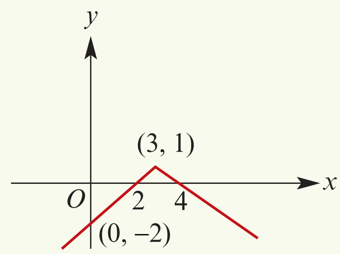
**b** $f(x) = -|x-3|+1 = \begin{cases} -(x-3)+1 & \text{if } x \geq 3 \\ -(3-x)+1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} -x+4 & \text{if } x \geq 3 \\ -2+x & \text{if } x < 3 \end{cases}$

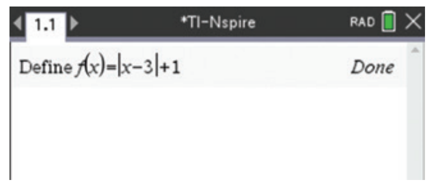
Range $= (-\infty, 1]$




## Using the TI-Nspire

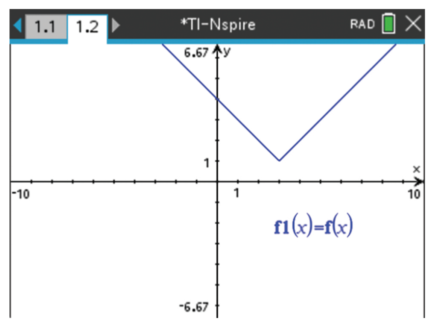
- Use menu > **Actions** > **Define** to define the function $f(x) = \text{abs}(x-3)+1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.



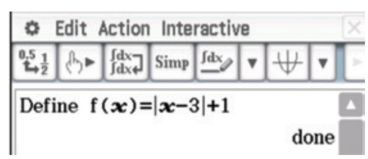

- Open a **Graphs** application (ctrl I > **Graphs**) and let $f1(x) = f(x)$.
- Press enter to obtain the graph.

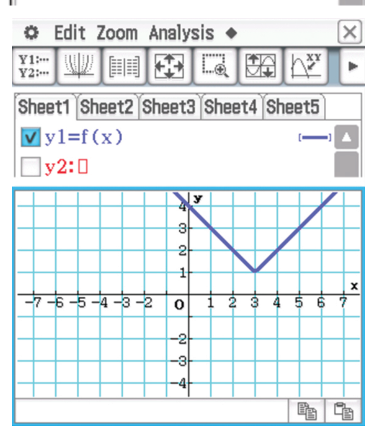
Note: The expression $\text{abs}(x-3)+1$ could have been entered directly for $f1(x)$.




## Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, define the function $f(x) = |x-3|+1$:
  - Select Define and f from the Math3 keyboard.
  - Complete the rule for $f$ as shown by using the modulus template from the Math1 keyboard.
  - Tap EXE.
- Open the **Graph & Table** application.
- Enter $f(x)$ in $y1$. Tick the box or tap EXE.
- Tap the graph icon to view the graph.

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## Functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$

If the graph of $y = f(x)$ is known, then we can sketch the graph of $y = |f(x)|$ using the following observation:

$$|f(x)| = f(x) \text{ if } f(x) \ge 0 \quad \text{and} \quad |f(x)| = -f(x) \text{ if } f(x) < 0$$

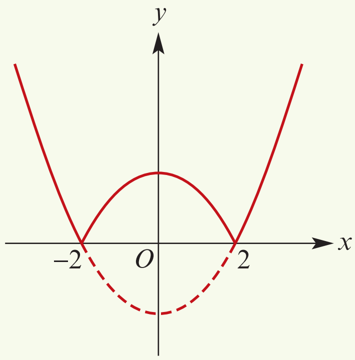
### Example 29

Sketch the graph of each of the following:

**a** $y = |x^2 - 4|$  **b** $y = |2^x - 1|$

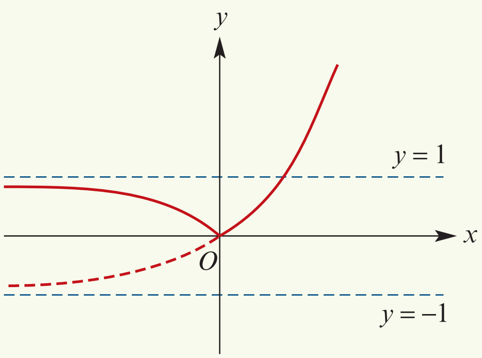
#### Solution

**a**




The graph of $y = x^2 - 4$ is drawn and the negative part reflected in the $x$-axis.

**b**




The graph of $y = 2^x - 1$ is drawn and the negative part reflected in the $x$-axis.

The graph of $y = f(|x|)$, for $x \in \mathbb{R}$, is sketched by reflecting the graph of $y = f(x)$, for $x \ge 0$, in the $y$-axis.

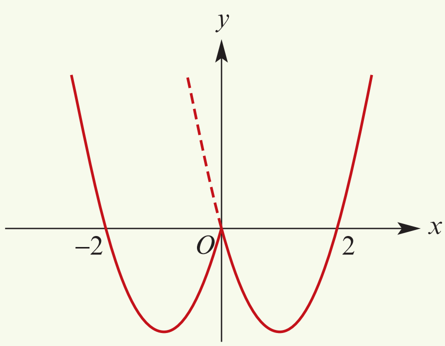
### Example 30

Sketch the graph of each of the following:

**a** $y = |x|^2 - 2|x|$  **b** $y = 2^{|x|}$

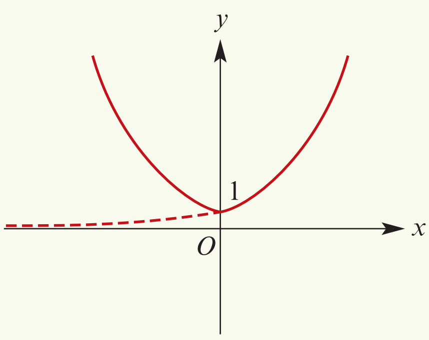
#### Solution

**a**




The graph of $y = x^2 - 2x$, $x \ge 0$, is reflected in the $y$-axis.

**b**




The graph of $y = 2^x$, $x \ge 0$, is reflected in the $y$-axis.

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Skill-sheet

## Exercise 1D

Example 26

**1** Evaluate each of the following:

**a** $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$
**d** $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

**2** Solve each of the following equations for $x$:

**a** $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$
**e** $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 27

**3** For each of the following, illustrate the set on a number line and represent the set using interval notation:

**a** $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$
**d** $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 28

**4** For each of the following functions, sketch the graph and state the range:

**a** $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$
**c** $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

**5** Solve each of the following inequalities, giving your answer using set notation:

**a** $\{ x : |x| \leq 5 \}$ **b** $\{ x : |x| \geq 2 \}$ **c** $\{ x : |2x - 3| \leq 1 \}$
**d** $\{ x : |5x - 2| < 3 \}$ **e** $\{ x : |-x + 3| \geq 7 \}$ **f** $\{ x : |-x + 2| \leq 1 \}$

**6** Solve each of the following for $x$:

**a** $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

Example 29

**7** Sketch the graph of each of the following:

**a** $y = |x^2 - 9|$ **b** $y = |3^x - 3|$ **c** $y = |x^2 - x - 12|$
**d** $y = |x^2 - 3x - 40|$ **e** $y = |x^2 - 2x - 8|$ **f** $y = |2^x - 4|$

Example 30

**8** Sketch the graph of each of the following:

**a** $y = |x|^2 - 4|x|$ **b** $y = 3^{|x|}$ **c** $y = |x|^2 - 7|x| + 12$
**d** $y = |x|^2 - |x| - 12$ **e** $y = |x|^2 + |x| - 12$ **f** $y = -3^{|x|} + 1$

**9** If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of $a$ and $b$.

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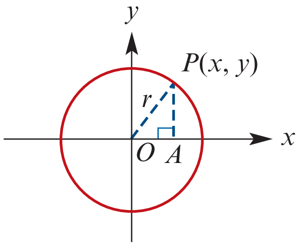
# 1E Circles

Consider a circle with centre at the origin and radius $r$.




If a point with coordinates $(x, y)$ lies on the circle, then Pythagoras' theorem gives

$$x^2 + y^2 = r^2$$

The converse is also true. That is, a point with coordinates $(x, y)$ such that $x^2 + y^2 = r^2$ lies on the circle.

**Cartesian equation of a circle**

The circle with centre $(h, k)$ and radius $r$ is the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

Note: This circle is obtained from the circle with equation $x^2 + y^2 = r^2$ by the translation defined by $(x, y) \to (x + h, y + k)$.

**Example 31**

Sketch the graph of the circle with centre $(-2, 5)$ and radius 2, and state the Cartesian equation for this circle.

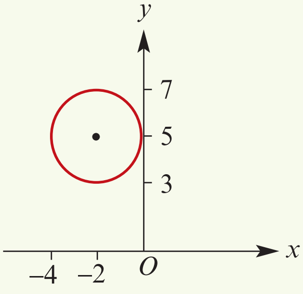
**Solution**

The equation is

$$(x + 2)^2 + (y - 5)^2 = 4$$

which may also be written as

$$x^2 + y^2 + 4x - 10y + 25 = 0$$




The equation $x^2 + y^2 + 4x - 10y + 25 = 0$ can be 'unsimplified' by completing the square:

$$x^2 + y^2 + 4x - 10y + 25 = 0$$
$$x^2 + 4x + 4 + y^2 - 10y + 25 + 25 = 29$$
$$(x + 2)^2 + (y - 5)^2 = 4$$

This suggests a general form of the equation of a circle:

$$x^2 + y^2 + Dx + Ey + F = 0$$

Completing the square gives

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} + F = \frac{D^2 + E^2}{4}$$

i.e.
$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}$$

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- If $D^2 + E^2 - 4F > 0$, then this equation represents a circle.
- If $D^2 + E^2 - 4F = 0$, then this equation represents one point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$.
- If $D^2 + E^2 - 4F < 0$, then this equation has no representation in the Cartesian plane.

### Example 32

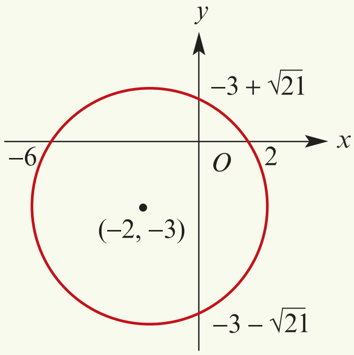
Sketch the graph of $x^2 + y^2 + 4x + 6y - 12 = 0$. State the coordinates of the centre and the radius.

**Solution**

Complete the square in both $x$ and $y$:

$$x^2 + y^2 + 4x + 6y - 12 = 0$$
$$x^2 + 4x + 4 + y^2 + 6y + 9 - 12 = 13$$
$$(x+2)^2 + (y+3)^2 = 25$$

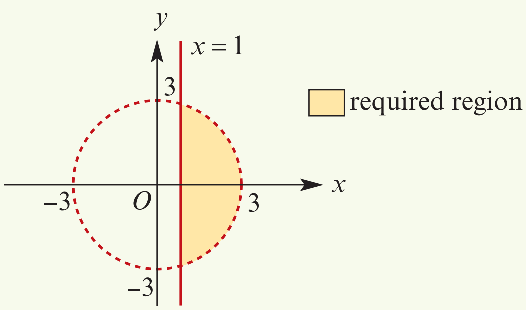
The circle has centre $(-2, -3)$ and radius 5.




### Example 33

Sketch a graph of the region of the plane such that $x^2 + y^2 < 9$ and $x \geq 1$.

**Solution**




## Exercise 1E

Example 31

**1** For each of the following, find the equation of the circle with the given centre and radius:

**a** centre $(2, 3)$; radius 1
**b** centre $(-3, 4)$; radius 5
**c** centre $(0, -5)$; radius 5
**d** centre $(3, 0)$; radius $\sqrt{2}$

Example 32

**2** Find the radius and the coordinates of the centre of the circle with equation:

**a** $x^2 + y^2 + 4x - 6y + 12 = 0$
**b** $x^2 + y^2 - 2x - 4y + 1 = 0$
**c** $x^2 + y^2 - 3x = 0$
**d** $x^2 + y^2 + 4x - 10y + 25 = 0$

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**3** Sketch the graph of each of the following:

**a** $2x^2 + 2y^2 + x + y = 0$

**b** $x^2 + y^2 + 3x - 4y = 6$

**c** $x^2 + y^2 + 8x - 10y + 16 = 0$

**d** $x^2 + y^2 - 8x - 10y + 16 = 0$

**e** $2x^2 + 2y^2 - 8x + 5y + 10 = 0$

**f** $3x^2 + 3y^2 + 6x - 9y = 100$

Example 33

**4** For each of the following, sketch the graph of the specified region of the plane:

**a** $x^2 + y^2 \le 16$

**b** $x^2 + y^2 \ge 9$

**c** $(x - 2)^2 + (y - 2)^2 < 4$

**d** $(x - 3)^2 + (y + 2)^2 > 16$

**e** $x^2 + y^2 \le 16$ and $x \le 2$

**f** $x^2 + y^2 \le 9$ and $y \ge -1$

**5** The points $(8, 4)$ and $(2, 2)$ are the ends of a diameter of a circle. Find the coordinates of the centre and the radius of the circle.

**6** Find the equation of the circle with centre $(2, -3)$ that touches the $x$-axis.

**7** Find the equation of the circle that passes through $(3, 1)$, $(8, 2)$ and $(2, 6)$.

**8** Consider the circles with equations

$$4x^2 + 4y^2 - 60x - 76y + 536 = 0 \quad \text{and} \quad x^2 + y^2 - 10x - 14y + 49 = 0$$

**a** Find the radius and the coordinates of the centre of each circle.

**b** Find the coordinates of the points of intersection of the two circles.

**9** Find the coordinates of the points of intersection of the circle with equation $x^2 + y^2 = 25$ and the line with equation:

**a** $y = x$

**b** $y = 2x$

# 1F Ellipses and hyperbolas

Ellipses and hyperbolas will arise in our study of vector calculus in Chapter 13. In this section, we revise sketching graphs of ellipses and hyperbolas from their Cartesian equations.

## Ellipses

For positive constants $a$ and $b$, the curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is obtained from the unit circle $x^2 + y^2 = 1$ by applying the following dilations:

- a dilation of factor $a$ from the $y$-axis, i.e. $(x, y) \to (ax, y)$
- a dilation of factor $b$ from the $x$-axis, i.e. $(x, y) \to (x, by)$.

The result is the transformation $(x, y) \to (ax, by)$.

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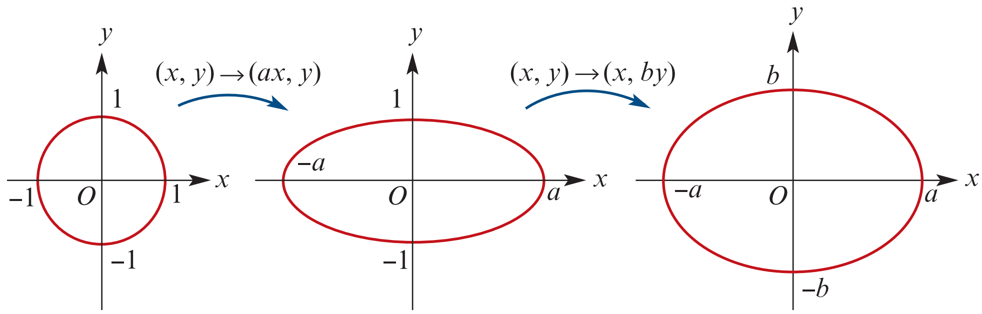


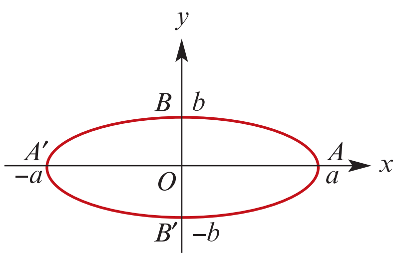

The curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse centred at the origin with $x$-axis intercepts at $(-a, 0)$ and $(a, 0)$ and with $y$-axis intercepts at $(0, -b)$ and $(0, b)$.

If $a = b$, then the ellipse is a circle centred at the origin with radius $a$.

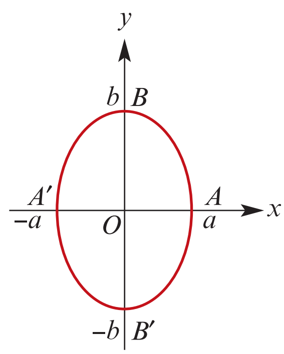
**Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$**



$AA'$ is the major axis

$BB'$ is the minor axis

**Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b > a$**



$AA'$ is the minor axis

$BB'$ is the major axis

### Cartesian equation of an ellipse

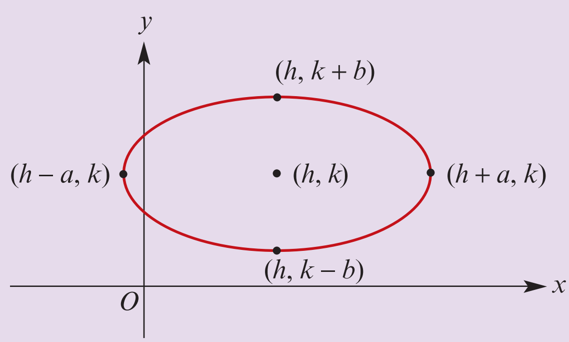
The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse with centre $(h, k)$. It is obtained from the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by the translation $(x, y) \to (x + h, y + k)$.

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## Example 34

Sketch the graph of each of the following ellipses. Give the coordinates of the centre and the axis intercepts.

**a** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

**b** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

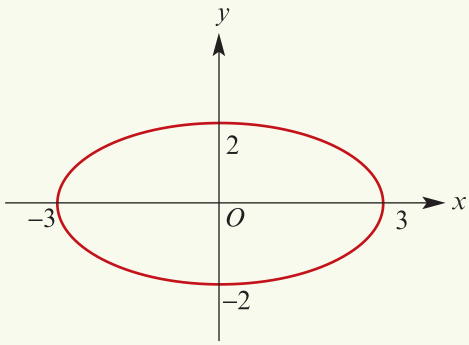
**c** $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

**d** $3x^2 + 24x + y^2 + 36 = 0$

### Solution

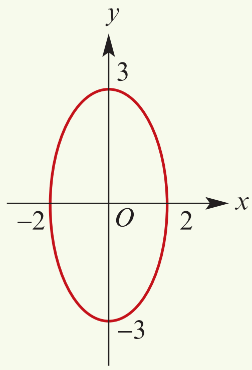
**a** Centre $(0, 0)$

Axis intercepts $(\pm 3, 0)$ and $(0, \pm 2)$




**b** Centre $(0, 0)$

Axis intercepts $(\pm 2, 0)$ and $(0, \pm 3)$




**c** Centre $(2, 3)$

*y*-axis intercepts

When $x = 0$: $\dfrac{4}{9} + \dfrac{(y-3)^2}{16} = 1$

$$\frac{(y-3)^2}{16} = \frac{5}{9}$$

$$(y-3)^2 = \frac{16 \times 5}{9}$$

$$\therefore \quad y = 3 \pm \frac{4\sqrt{5}}{3}$$

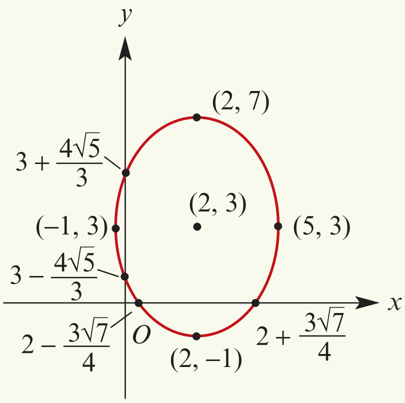
*x*-axis intercepts

When $y = 0$: $\dfrac{(x-2)^2}{9} + \dfrac{9}{16} = 1$

$$\frac{(x-2)^2}{9} = \frac{7}{16}$$

$$(x-2)^2 = \frac{9 \times 7}{16}$$

$$\therefore \quad x = 2 \pm \frac{3\sqrt{7}}{4}$$

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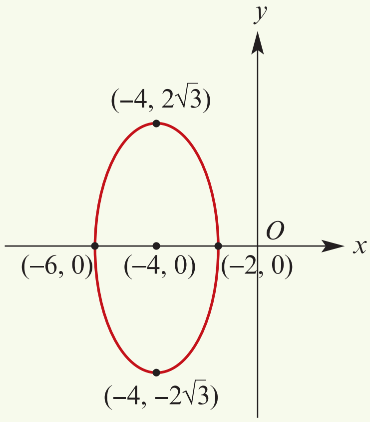
**d** Completing the square:

$$3x^2 + 24x + y^2 + 36 = 0$$
$$3(x^2 + 8x + 16) + y^2 + 36 - 48 = 0$$
$$3(x+4)^2 + y^2 = 12$$

i.e. $$\frac{(x+4)^2}{4} + \frac{y^2}{12} = 1$$

Centre $(-4, 0)$

Axis intercepts $(-6, 0)$ and $(-2, 0)$




Given an equation of the form

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

where both $A$ and $B$ are positive, there are three possibilities for the corresponding graph. The graph may be an ellipse (which includes the special case where the graph is a circle), the graph may be a single point, or there may be no pairs $(x, y)$ that satisfy the equation.

## Hyperbolas

The curve with equation

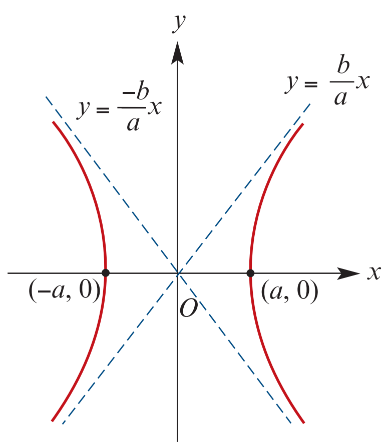
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a hyperbola centred at the origin with axis intercepts $(a, 0)$ and $(-a, 0)$.

The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

To see why this should be the case, we rearrange the equation of the hyperbola as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$
$$\therefore \quad y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$




As $x \to \pm\infty$, we have $\frac{a^2}{x^2} \to 0$. This suggests that

$$y^2 \to \frac{b^2x^2}{a^2}$$

i.e. $$y \to \pm\frac{bx}{a}$$

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### Cartesian equation of a hyperbola

The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola with centre $(h, k)$. The asymptotes are

$$y - k = \pm\frac{b}{a}(x - h)$$

**Note:** This hyperbola is obtained from the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the translation defined by $(x, y) \to (x + h, y + k)$.

## Example 35

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

**a** $\frac{x^2}{9} - \frac{y^2}{4} = 1$

**b** $\frac{y^2}{9} - \frac{x^2}{4} = 1$

**c** $(x-1)^2 - (y+2)^2 = 1$

**d** $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

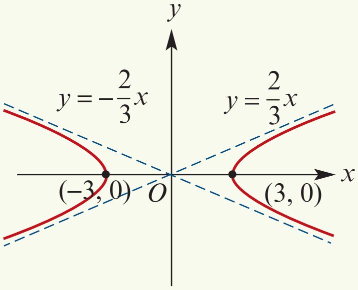
### Solution

**a** Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The $x$-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.



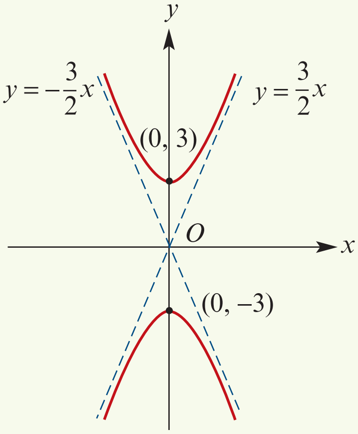

**b** Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

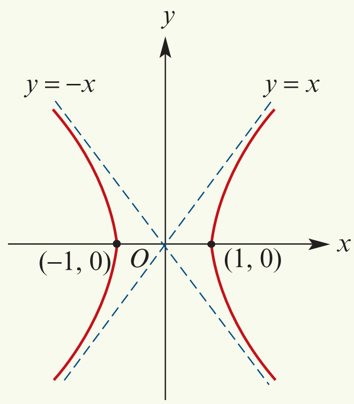
The $y$-axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.



**c** First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This is called a **rectangular hyperbola**, as its asymptotes are perpendicular.




Now to sketch the graph of

$$(x-1)^2 - (y+2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

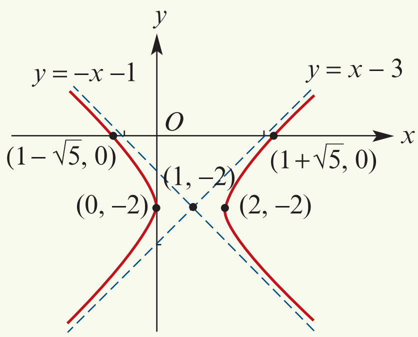
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

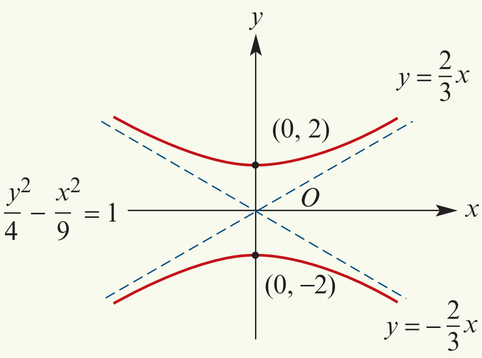
If $x = 0$, then $y = -2$.

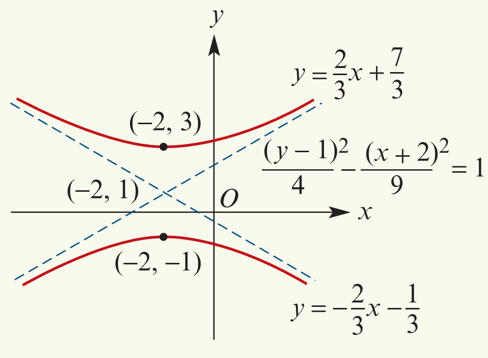
If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.




**d** The graph of $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ through the translation $(x, y) \to (x - 2, y + 1)$. Its centre will be $(-2, 1)$.




The axis intercepts are $\left(0, 1 \pm \frac{2\sqrt{13}}{3}\right)$.

Note: The hyperbolas $\frac{y^2}{4} - \frac{x^2}{9} = 1$ and $\frac{x^2}{9} - \frac{y^2}{4} = 1$ have the same asymptotes; they are called **conjugate hyperbolas**.

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Skill-sheet

## Exercise 1F

Example 34

**1** Sketch the graph of each of the following. Label the axis intercepts and state the coordinates of the centre.

**a** $\frac{x^2}{9} + \frac{y^2}{16} = 1$

**b** $25x^2 + 16y^2 = 400$

**c** $\frac{(x-4)^2}{9} + \frac{(y-1)^2}{16} = 1$

**d** $x^2 + \frac{(y-2)^2}{9} = 1$

**e** $9x^2 + 25y^2 - 54x - 100y = 44$

**f** $9x^2 + 25y^2 = 225$

**g** $5x^2 + 9y^2 + 20x - 18y - 16 = 0$

**h** $16x^2 + 25y^2 - 32x + 100y - 284 = 0$

**i** $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

**j** $2(x-2)^2 + 4(y-1)^2 = 16$

Example 35

**2** Sketch the graph of each of the following. Label the axis intercepts and give the equations of the asymptotes.

**a** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

**b** $\frac{y^2}{16} - \frac{x^2}{9} = 1$

**c** $x^2 - y^2 = 4$

**d** $2x^2 - y^2 = 4$

**e** $x^2 - 4y^2 - 4x - 8y - 16 = 0$

**f** $9x^2 - 25y^2 - 90x + 150y = 225$

**g** $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

**h** $4x^2 - 8x - y^2 + 2y = 0$

**i** $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

**j** $25x^2 - 16y^2 = 400$

**3** Find the coordinates of the points of intersection of $y = \frac{1}{2}x$ with:

**a** $x^2 - y^2 = 1$

**b** $\frac{x^2}{4} + y^2 = 1$

**4** Show that there is no intersection point of the line $y = x + 5$ with the ellipse $x^2 + \frac{y^2}{4} = 1$.

**5** Let $a, b > 0$. Prove that the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ intersect on the vertices of a square.

**6** Find the coordinates of the points of intersection of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the line with equation $5x = 4y$.

**7** On the one set of axes, sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 9$.

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# 1G Parametric equations

In Chapter 13, we will study motion along a curve. A **parameter** (usually $t$ representing time) will be used to help describe these curves. In Chapter 5, we will use a parameter to describe lines in two- or three-dimensional space.

This section gives an introduction to parametric equations of curves in the plane.

## The unit circle

The unit circle can be expressed in Cartesian form as $\{(x, y) : x^2 + y^2 = 1\}$. We have seen in Section 1A that the unit circle can also be expressed as

$$\{(x, y) : x = \cos t \text{ and } y = \sin t, \text{ for some } t \in \mathbb{R}\}$$

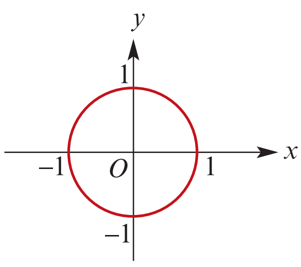
The set notation is often omitted, and we can describe the unit circle by the equations

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for } t \in \mathbb{R}$$

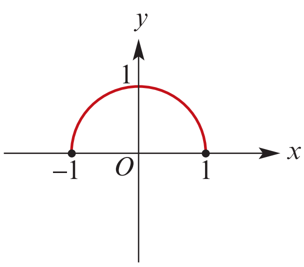
These are the **parametric equations** for the unit circle.

We still obtain the entire unit circle if we restrict the values of $t$ to the interval $[0, 2\pi]$. The following three diagrams illustrate the graphs obtained from the parametric equations $x = \cos t$ and $y = \sin t$ for three different sets of values of $t$.

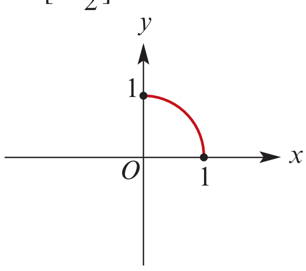
$t \in [0, 2\pi]$




$t \in [0, \pi]$




$t \in \left[0, \frac{\pi}{2}\right]$




## Circles

**Parametric equations for a circle centred at the origin**

The circle with centre the origin and radius $a$ is described by the parametric equations

$$x = a\cos t \quad \text{and} \quad y = a\sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

Note: To obtain the Cartesian equation, first rearrange the parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{a} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \cos^2 t + \sin^2 t = 1$$

This equation can be written as $x^2 + y^2 = a^2$, which is the Cartesian equation of the circle with centre the origin and radius $a$.

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The domain and range of the circle can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.
- **Range** The range of the function with rule $y = a\sin t$ is $[-a, a]$.
  Hence the range of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.

### Example 36

A circle is defined by the parametric equations

$$x = 2 + 3\cos\theta \quad \text{and} \quad y = 1 + 3\sin\theta \quad \text{for } \theta \in [0, 2\pi]$$

Find the Cartesian equation of the circle, and state the domain and range of this relation.

**Solution**

**Domain** The range of the function with rule $x = 2 + 3\cos\theta$ is $[-1, 5]$. Hence the domain of the corresponding Cartesian relation is $[-1, 5]$.

**Range** The range of the function with rule $y = 1 + 3\sin\theta$ is $[-2, 4]$. Hence the range of the corresponding Cartesian relation is $[-2, 4]$.

**Cartesian equation**

Rewrite the parametric equations as

$$\frac{x-2}{3} = \cos\theta \quad \text{and} \quad \frac{y-1}{3} = \sin\theta$$

Square both sides of each of these equations and add:

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{9} = \cos^2\theta + \sin^2\theta = 1$$

i.e. $(x-2)^2 + (y-1)^2 = 9$

**Parametric equations for a circle**

The circle with centre $(h, k)$ and radius $a$ is described by the parametric equations

$$x = h + a\cos t \quad \text{and} \quad y = k + a\sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

## Parametric equations in general

A **parametric curve** in the plane is defined by a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable $t$ is called the **parameter**. Each value of $t$ gives a point $(f(t), g(t))$ in the plane. The set of all such points will be a curve in the plane.

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Note: If $x = f(t)$ and $y = g(t)$ are parametric equations for a curve $C$ and you eliminate the parameter $t$ between the two equations, then each point of the curve $C$ lies on the curve represented by the resulting Cartesian equation.

## Example 37

A curve is defined parametrically by the equations

$$x = at^2 \quad \text{and} \quad y = 2at \quad \text{for } t \in \mathbb{R}$$

where $a$ is a positive constant. Find:

**a** the Cartesian equation of the curve

**b** the equation of the line passing through the points where $t = 1$ and $t = -2$

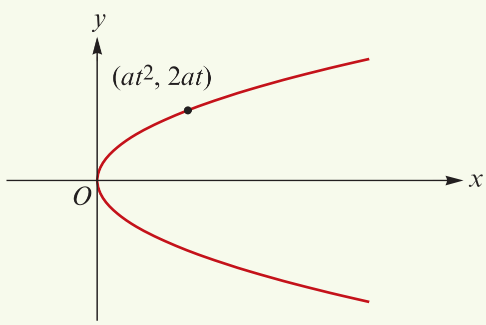
**c** the length of the chord joining the points where $t = 1$ and $t = -2$.

### Solution

**a** The second equation gives $t = \frac{y}{2a}$.

Substitute this into the first equation:

$$x = at^2 = a\left(\frac{y}{2a}\right)^2 = a\left(\frac{y^2}{4a^2}\right) = \frac{y^2}{4a}$$




This can be written as $y^2 = 4ax$.

**b** At $t = 1$, $x = a$ and $y = 2a$. This is the point $(a, 2a)$.

At $t = -2$, $x = 4a$ and $y = -4a$. This is the point $(4a, -4a)$.

The gradient of the line is

$$m = \frac{2a - (-4a)}{a - 4a} = \frac{6a}{-3a} = -2$$

Therefore the equation of the line is

$$y - 2a = -2(x - a)$$

which simplifies to $y = -2x + 4a$.

**c** The chord joining $(a, 2a)$ and $(4a, -4a)$ has length

$$\sqrt{(a - 4a)^2 + (2a - (-4a))^2} = \sqrt{9a^2 + 36a^2} = \sqrt{45a^2} = 3\sqrt{5}\,a \qquad \text{(since } a > 0\text{)}$$

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## Ellipses

**Parametric equations for an ellipse**

The ellipse with the Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$

The entire ellipse is obtained by taking $t \in [0, 2\pi]$.

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

The domain and range of the ellipse can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-a, a]$.
- **Range** The range of the function with rule $y = b\sin t$ is $[-b, b]$.
  Hence the range of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-b, b]$.

### Example 38

Find the Cartesian equation of the curve with parametric equations

$$x = 3 + 3\sin t \quad \text{and} \quad y = 2 - 2\cos t \quad \text{for } t \in \mathbb{R}$$

and describe the graph.

**Solution**

We can rearrange the two equations as

$$\frac{x-3}{3} = \sin t \quad \text{and} \quad \frac{2-y}{2} = \cos t$$

Now square both sides of each equation and add:

$$\frac{(x-3)^2}{9} + \frac{(2-y)^2}{4} = \sin^2 t + \cos^2 t = 1$$

Since $(2-y)^2 = (y-2)^2$, this equation can be written more neatly as

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

This is the equation of an ellipse with centre $(3, 2)$ and axis intercepts at $(3, 0)$ and $(0, 2)$.

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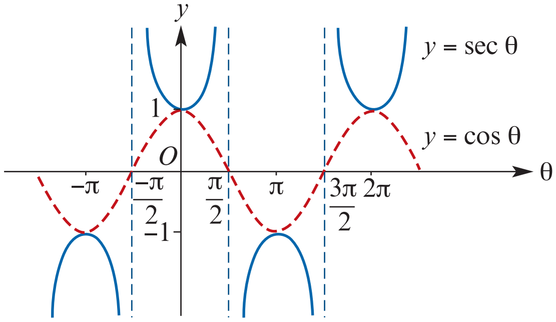
## Hyperbolas

In order to give parametric equations for hyperbolas, we will be using the **secant function**, which is defined by

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{if } \cos\theta \neq 0$$

The graphs of $y = \sec\theta$ and $y = \cos\theta$ are shown here on the same set of axes. The secant function is studied further in Chapter 3.




We will also use an alternative form of the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Dividing both sides by $\cos^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$

We will use this identity in the form

$$\sec^2\theta - \tan^2\theta = 1$$

### Parametric equations for a hyperbola

The hyperbola with the Cartesian equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\sec t \quad \text{and} \quad y = b\tan t \quad \text{for } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

Square and subtract these equations to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

The domain and range of the hyperbola can be determined from the parametric equations.

- **Domain** There are two cases, giving the left and right branches of the hyperbola:
  - For $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $[a, \infty)$.
    The domain $[a, \infty)$ gives the right branch of the hyperbola.
  - For $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $(-\infty, a]$.
    The domain $(-\infty, a]$ gives the left branch of the hyperbola.
- **Range** For both sections of the domain, the range of the function with rule $y = b\tan t$ is $\mathbb{R}$.

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**Example 39**

Find the Cartesian equation of the curve with parametric equations

$$x = 3\sec t \quad \text{and} \quad y = 4\tan t \quad \text{for } t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Describe the curve.

**Solution**

Rearrange the two equations:

$$\frac{x}{3} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Square both sides of each equation and subtract:

$$\frac{x^2}{9} - \frac{y^2}{16} = \sec^2 t - \tan^2 t = 1$$

The Cartesian equation of the curve is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

The range of the function with rule $x = 3\sec t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is $(-\infty, -3]$. Hence the domain for the graph is $(-\infty, -3]$.

The curve is the left branch of a hyperbola centred at the origin with $x$-axis intercept at $(-3, 0)$. The equations of the asymptotes are $y = \frac{4x}{3}$ and $y = -\frac{4x}{3}$.

## Finding parametric equations for a curve

When converting from a Cartesian equation to a pair of parametric equations, there are many different possible choices.

**Example 40**

Give parametric equations for each of the following:

**a** $x^2 + y^2 = 9$

**b** $\frac{x^2}{16} + \frac{y^2}{4} = 1$

**c** $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

**Solution**

**a** One possible solution is $x = 3\cos t$ and $y = 3\sin t$ for $t \in [0, 2\pi]$.

Another solution is $x = -3\cos(2t)$ and $y = 3\sin(2t)$ for $t \in [0, \pi]$.

Yet another solution is $x = 3\sin t$ and $y = 3\cos t$ for $t \in \mathbb{R}$.

**b** One possible solution is $x = 4\cos t$ and $y = 2\sin t$.

**c** One possible solution is $x - 1 = 3\sec t$ and $y + 1 = 2\tan t$.

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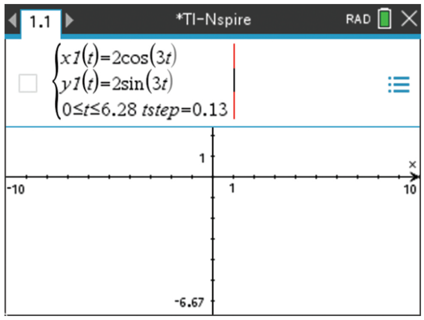
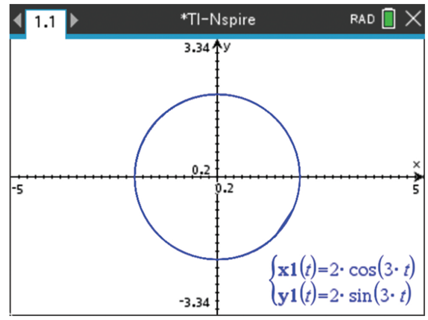
## Using the TI-Nspire

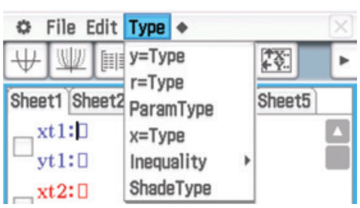
- Open a **Graphs** application (on > **New** > **Add Graphs**).
- Use menu > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.



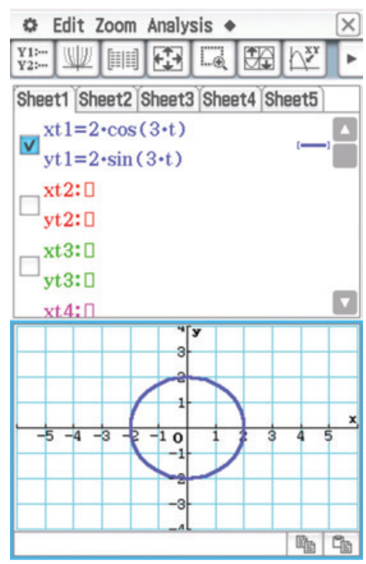




## Using the Casio ClassPad

- Open the **Graph & Table** application .
- From the toolbar, select **Type** > **ParamType**.




- Use the Trig keyboard to enter the equations as shown on the right.
- Tick the box and tap .
- Use to adjust the window.




Skill-sheet

## Exercise 1G

Example 36

**1** Find the Cartesian equation of the curve with parametric equations $x = 2\cos(3t)$ and $y = 2\sin(3t)$, and determine the domain and range of the corresponding relation.

Example 37

**2** A curve is defined parametrically by the equations $x = 4t^2$ and $y = 8t$ for $t \in \mathbb{R}$. Find:

**a** the Cartesian equation of the curve

**b** the equation of the line passing through the points where $t = 1$ and $t = -1$

**c** the length of the chord joining the points where $t = 1$ and $t = -3$.

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Example 38

**3** Find the Cartesian equation of the curve with parametric equations $x = 2 + 3\sin t$ and $y = 3 - 2\cos t$ for $t \in \mathbb{R}$, and describe the graph.

Example 39

**4** Find the Cartesian equation of the curve with parametric equations $x = 2\sec t$ and $y = 3\tan t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and describe the curve.

**5** Find the corresponding Cartesian equation for each pair of parametric equations:

**a** $x = 4\cos(2t)$ and $y = 4\sin(2t)$
**b** $x = 2\sin(2t)$ and $y = 2\cos(2t)$
**c** $x = 4\cos t$ and $y = 3\sin t$
**d** $x = 4\sin t$ and $y = 3\cos t$
**e** $x = 2\tan(2t)$ and $y = 3\sec(2t)$
**f** $x = 1 - t$ and $y = t^2 - 4$
**g** $x = t + 2$ and $y = \frac{1}{t}$
**h** $x = t^2 - 1$ and $y = t^2 + 1$
**i** $x = t - \frac{1}{t}$ and $y = 2\left(t + \frac{1}{t}\right)$

**6** For each of the following pairs of parametric equations, determine the Cartesian equation of the curve and sketch its graph:

**a** $x = \sec t,\ y = \tan t,\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
**b** $x = 3\cos(2t),\ y = -4\sin(2t)$
**c** $x = 3 - 3\cos t,\ y = 2 + 2\sin t$
**d** $x = 3\sin t,\ y = 4\cos t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
**e** $x = \sec t,\ y = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
**f** $x = 1 - \sec(2t),\ y = 1 + \tan(2t),\ t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

**7** A circle is defined by the parametric equations

$$x = 2\cos(2t) \quad \text{and} \quad y = -2\sin(2t) \quad \text{for } t \in \mathbb{R}$$

**a** Find the coordinates of the point $P$ on the circle where $t = \frac{4\pi}{3}$.
**b** Find the equation of the tangent to the circle at $P$.

Example 40

**8** Give parametric equations corresponding to each of the following:

**a** $x^2 + y^2 = 16$
**b** $\frac{x^2}{9} - \frac{y^2}{4} = 1$
**c** $(x - 1)^2 + (y + 2)^2 = 9$
**d** $\frac{(x - 1)^2}{9} + \frac{(y + 3)^2}{4} = 9$

**9** A circle has centre $(1, 3)$ and radius 2. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where $a$, $b$, $c$ and $d$ are positive constants, state the values of $a$, $b$, $c$ and $d$.

**10** An ellipse has $x$-axis intercepts $(-4, 0)$ and $(4, 0)$ and $y$-axis intercepts $(0, 3)$ and $(0, -3)$. State a possible pair of parametric equations for this ellipse.

**11** The circle with parametric equations $x = 2\cos(2t)$ and $y = 2\sin(2t)$ is dilated by a factor of 3 from the $x$-axis. For the image curve, state:

**a** a possible pair of parametric equations
**b** the Cartesian equation.

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**12** The ellipse with parametric equations $x = 3 - 2\cos\left(\frac{t}{2}\right)$ and $y = 4 + 3\sin\left(\frac{t}{2}\right)$ is translated 3 units in the negative direction of the $x$-axis and 2 units in the negative direction of the $y$-axis. For the image curve, state:

**a** a possible pair of parametric equations **b** the Cartesian equation.

**13** Sketch the graph of the curve with parametric equations $x = 2 + 3\sin(2\pi t)$ and $y = 4 + 2\cos(2\pi t)$ for:

**a** $t \in [0, \frac{1}{4}]$ **b** $t \in [0, \frac{1}{2}]$ **c** $t \in [0, \frac{3}{2}]$

For each of these graphs, state the domain and range.

# 1H Algorithms and pseudocode

An **algorithm** is a finite, unambiguous sequence of instructions for performing a specific task. An algorithm can be described using step-by-step instructions, illustrated by a flowchart, or written out in pseudocode.

You have seen many examples of algorithms in Year 11 and you will meet several new algorithms throughout this book. This section gives a summary of writing algorithms in pseudocode.

Note: The Interactive Textbook includes online appendices that provide an introduction to coding using the language *Python* and also to coding using the TI-Nspire and the Casio ClassPad.

## Assigning values to variables

A **variable** is a string of one or more letters that acts as a placeholder that can be assigned different values. For example, the notation

$$x \leftarrow 3$$

means 'assign the value 3 to the variable $x$'.

## Controlling the flow of steps

The steps of an algorithm are typically carried out one after the other. However, there are constructs that allow us to control the flow of steps.

- **If–then blocks** This construct provides a means of making decisions within an algorithm. Certain instructions are only followed if a condition is satisfied.

```
if condition then
    follow these instructions
end if
```

- **For loops** This construct provides a means of repeatedly executing the same set of instructions in a controlled way. In the template on the right, this is achieved by performing one iteration for each value of $i$ in the sequence $1, 2, 3, \dots, n$.

```
for i from 1 to n
    follow these instructions
end for
```

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- **While loops** This construct provides another means of repeatedly executing the same set of instructions in a controlled way. This is achieved by performing iterations indefinitely, as long as some condition remains true.

```
while condition
    follow these instructions
end while
```

In the following example, we construct a table of values to demonstrate the algorithm. This is called a **desk check**. In general, we carry out a desk check of an algorithm by carefully following the algorithm step by step, and constructing a table of the values of all the variables after each step.

### Example 41

Consider the sequence defined by the rule

$$x_{n+1} = 3x_n - 2, \qquad \text{where } x_1 = 3$$

Write an algorithm that will determine the smallest value of $n$ for which $x_n > 1000$. Show a desk check to test the operation of the algorithm.

**Solution**

We use a `while` loop, since we don't know how many iterations will be required.

The variable $x$ is used for the current term of the sequence, and the variable $n$ is used to keep track of the number of iterations.

```
n ← 1
x ← 3
while x ≤ 1000
    n ← n + 1
    x ← 3x − 2
end while
print n
```

| $n$ | $x$ |
|---|---|
| 1 | 3 |
| 2 | 7 |
| 3 | 19 |
| 4 | 55 |
| 5 | 163 |
| 6 | 487 |
| 7 | 1459 |

Note: The output is 7.

## Functions

A **function** takes one or more input values and returns an output value. Functions can be defined and then used in other algorithms.

```
define function(input):
    follow these instructions
    return output
```

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### Example 42

Construct a function that inputs a natural number $n$ and outputs the value of

$$n! = 1 \times 2 \times \cdots \times n$$

**Solution**

```
define factorial(n):
   product ← 1
   for i from 1 to n
       product ← product × i
   end for
   return product
```

For example, calling *factorial*(4) will return the value 24.

## Lists

In programming languages, a finite sequence is often called a **list**. We will write lists using square brackets. For example, we can define a list $A$ by

$$A \leftarrow [2, 3, 5, 7, 11]$$

The notation $A[n]$ refers to the $n$th entry of the list. So $A[1] = 2$ and $A[5] = 11$.

We can add an entry to the end of a list using append. For example, the instruction

```
append 9 to A
```

would result in $A = [2, 3, 5, 7, 11, 9]$.

### Example 43

Write a function that returns a list of the first $n$ square numbers for a given natural number $n$.

**Solution**

```
define squares(n):
   A ← [ ]
   for i from 1 to n
       append i^2 to A
   end for
   return A
```

For example, calling *squares*(5) will return the list $[1, 4, 9, 16, 25]$.

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## Nested loops

The next example illustrates how we can use loops within loops.

### Example 44

Using pseudocode, write an algorithm to find the positive integer solutions of the equation

$$11x + 19y + 13z = 200$$

**Solution**

We use three loops to run through all the possible positive integer values of $x$, $y$ and $z$. We first note that

$$200 \div 11 \approx 18.2, \qquad 200 \div 19 \approx 10.5, \qquad 200 \div 13 \approx 15.4$$

Therefore we know that we will find all the solutions from the following nest of three loops.

```
for x from 1 to 18
    for y from 1 to 10
        for z from 1 to 15
            if 11x + 19y + 13z = 200 then
                print (x, y, z)
            end if
        end for
    end for
end for
```

This algorithm prints the five solutions $(2, 8, 2)$, $(3, 4, 7)$, $(6, 5, 3)$, $(7, 1, 8)$ and $(10, 2, 4)$.

Skill-sheet

## Exercise 1H

Example 41

**1** Consider the sequence defined by the rule

$$x_{n+1} = 2x_n + 3, \qquad \text{where } x_1 = 3$$

Write an algorithm that will determine the smallest value of $n$ for which $x_n > 100$. Show a desk check to test the operation of the algorithm.

Example 42

**2** Construct a function that inputs a natural number $n$ and outputs the product of the first $n$ even natural numbers.

Example 43

**3** Write a function that returns a list of the first $n$ powers of 2 for a given natural number $n$.

Example 44

**4** For each of the following, use pseudocode to describe an algorithm that will find all the positive integer solutions of the equation:

**a** $x^2 + y^2 + z^2 = 500$ **b** $x^3 + y^3 + z^3 = 1\,000\,000$

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**5** The sine function can be given by an infinite sum:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The following pseudocode function evaluates the sum of the first $n$ terms for a given value of $x$. (The code uses the factorial function from Example 42.)

```
define sinsum(x, n):
  sum ← 0
  for k from 1 to n
      sum ← sum + (−1)^(k+1) × x^(2k−1) / factorial(2k − 1)
  end for
  return sum
```

**a** Perform a desk check to evaluate:

**i** $sinsum(0.1, 4)$ **ii** $sinsum(1, 4)$ **iii** $sinsum(2, 4)$

**b** Compare the values found in part **a** with the values of $\sin 0.1$, $\sin 1$ and $\sin 2$.

**c** Using a device, investigate the function $sinsum(x, n)$ for a range of input values.

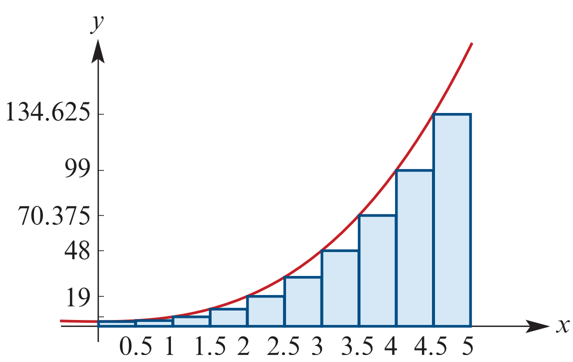
**6** The cosine function can be given by an infinite sum:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

**a** Modify the pseudocode function $sinsum(x, n)$ from the previous question to give approximations of the cosine function.

**b** Find an approximation for $\cos 2$ using the first four terms of the sum.

**7** Riemann sums The block of pseudocode on the right finds an approximation to the area under the curve $y = x^3 + 2x^2 + 3$ between $x = 0$ and $x = 5$. This is done by summing the areas of 10 rectangular strips, as shown in the diagram below.




```
define f(x):
  return x^3 + 2x^2 + 3

a ← 0
b ← 5
n ← 10
h ← (b − a)/n
left ← a
sum ← 0
for i from 1 to n
    strip ← f(left) × h
    sum ← sum + strip
    left ← left + h
end for
print sum
```

**a** Carry out a desk check for the algorithm.

**b** Modify the algorithm to use 50 trapezoidal strips.

Note: We will use similar algorithms for length, surface area and volume in Chapter 14.

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**8** **Newton's method** You met Newton's method in Mathematical Methods Units 1 & 2. We aim to find an approximate solution to an equation of the form $f(x) = 0$, where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. We start from an initial estimate $x = x_0$ and construct a sequence of approximations $x_1, x_2, x_3, \dots$ using the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{where } n = 0, 1, 2, \dots$$

The following algorithm uses this method for the equation $-x^3 + 5x^2 - 3x + 4 = 0$. The table shows a desk check of the algorithm.

```
define f(x):
   return −x^3 + 5x^2 − 3x + 4

define Df(x):
   return −3x^2 + 10x − 3

x ← 3.8
while |f(x)| > 10^−6
      x ← x − f(x)/Df(x)
      print x, f(x)
end while
```

| | $x$ | $f(x)$ |
|---|---|---|
| Initial | 3.8 | 9.928 |
| Pass 1 | 4.99326923 | −10.81199119 |
| Pass 2 | 4.60526316 | −1.44403339 |
| Pass 3 | 4.53507148 | −0.04308844 |
| Pass 4 | 4.53284468 | −0.00004266 |
| Pass 5 | 4.53284247 | 0.00000000 |

**a** Modify this algorithm for the equation $x^3 - 2 = 0$ and the initial estimate $x_0 = 2$. Carry out a desk check to determine an approximation of $2^{\frac{1}{3}}$.

**b** In Mathematical Methods Units 3 & 4, you will learn that if $f(x) = \sin x$, then $f'(x) = \cos x$. Use this fact and Newton's method to find an approximation of $\pi$. Start with the initial estimate $x_0 = 3$.

Note: We will use a similar method in Chapter 11 for the numerical solution of differential equations.

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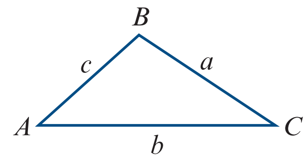
## Chapter summary

Assign-ment

Nrich

### Triangles

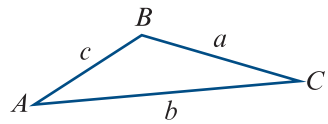
- Labelling triangles
  - Interior angles are denoted by uppercase letters.
  - The length of the side opposite an angle is denoted by the corresponding lowercase letter.




- Sine rule

  For triangle $ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

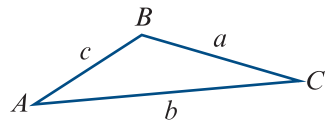



- Cosine rule

  For triangle $ABC$:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$




### Sequences and series

- The $n$th term of a sequence is denoted by $t_n$.
- A **recurrence relation** enables each subsequent term to be found from previous terms. A sequence specified in this way is said to be defined **recursively**.

  e.g. $t_1 = 1, \quad t_n = t_{n-1} + 2$

- A sequence may also be defined by a rule that is stated in terms of $n$.

  e.g. $t_n = 2n - 1$

- Arithmetic sequences and series
  - An **arithmetic sequence** has a rule of the form $t_n = a + (n-1)d$, where $a$ is the first term and $d$ is the **common difference** (i.e. $d = t_k - t_{k-1}$ for all $k > 1$).
  - The sum of the first $n$ terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

- Geometric sequences and series
  - A **geometric sequence** has a rule of the form $t_n = ar^{n-1}$, where $a$ is the first term and $r$ is the **common ratio** (i.e. $r = \frac{t_k}{t_{k-1}}$ for all $k > 1$).
  - For $r \neq 1$, the sum of the first $n$ terms of a geometric sequence is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

  - For $-1 < r < 1$, the sum $S_n$ approaches a limiting value as $n \to \infty$, and the series is said to be **convergent**. This limit is called the **sum to infinity** and is given by $S_\infty = \frac{a}{1 - r}$.

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Review

- **Recurrence relations of the form $t_n = rt_{n-1} + d$**
  Let $t_1, t_2, t_3, \ldots$ be a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r$ and $d$ are constants. Then the $n$th term of the sequence is given by
  $$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \qquad \text{(provided } r \neq 1\text{)}$$

## The modulus function

- The **modulus** or **absolute value** of a real number $x$ is
  $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
  For example: $|5| = 5$ and $|-5| = 5$.
- On the number line, the distance between two numbers $a$ and $b$ is given by $|a - b| = |b - a|$.
  For example: $|x - 2| < 5$ can be read as 'the distance of $x$ from 2 is less than 5'.

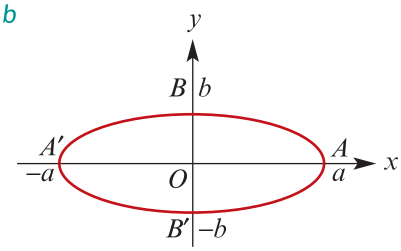
## Circles

- The circle with centre at the origin and radius $a$ has Cartesian equation $x^2 + y^2 = a^2$.
- The circle with centre $(h, k)$ and radius $a$ has equation $(x - h)^2 + (y - k)^2 = a^2$.

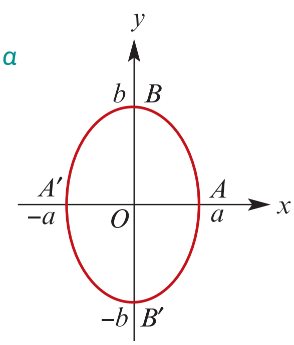
## Ellipses

- The curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse centred at the origin with axis intercepts $(\pm a, 0)$ and $(0, \pm b)$.

$a > b$



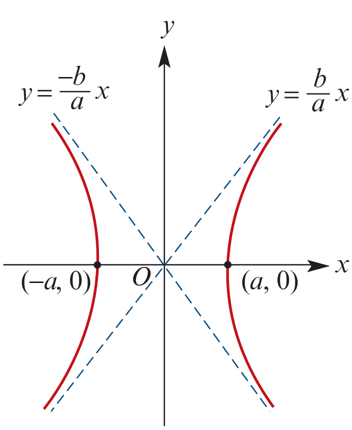

$b > a$




- The curve with equation $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ is an ellipse with centre $(h, k)$.

## Hyperbolas

- The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola centred at the origin.
  - The axis intercepts are $(\pm a, 0)$.
  - The asymptotes have equations $y = \pm\frac{b}{a}x$.

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- The curve with equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is a hyperbola with centre $(h, k)$. The asymptotes have equations $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.

### Parametric equations

- A **parametric curve** in the plane is defined by a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where $t$ is called the **parameter** of the curve.

- Parameterisations of familiar curves:

| | Cartesian equation | Parametric equations |
|---|---|---|
| Circle | $x^2 + y^2 = a^2$ | $x = a\cos t$ and $y = a\sin t$ |
| Ellipse | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $x = a\cos t$ and $y = b\sin t$ |
| Hyperbola | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $x = a\sec t$ and $y = b\tan t$ |

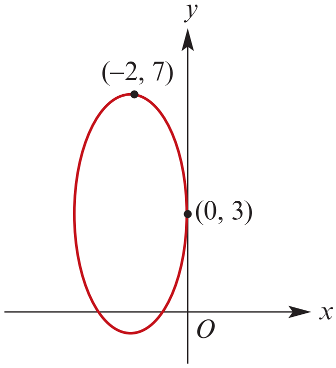
Note: To obtain the entire circle or the entire ellipse using these parametric equations, it suffices to take $t \in [0, 2\pi]$.

- Translations of parametric curves: The circle with equation $(x-h)^2 + (y-k)^2 = a^2$ can also be described by the parametric equations $x = h + a\cos t$ and $y = k + a\sin t$.

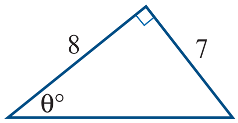
## Technology-free questions

**1** A sequence is defined recursively by $f_n = 5f_{n-1}$ and $f_0 = 1$. Find $f_n$ in terms of $n$.

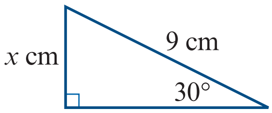
**2** Write down the equation of the ellipse shown.




**3** Find $\sin\theta°$.




**4** Find $x$.




**5** **a** Find the exact value of $\cos 315°$.

**b** Given that $\tan x° = \frac{3}{4}$ and $180 < x < 270$, find the exact value of $\cos x°$.

**c** Find an angle $A°$ (with $A \neq 330$) such that $\sin A° = \sin 330°$.

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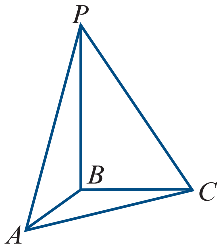
**6** $ABC$ is a horizontal right-angled triangle with the right angle at $B$. The point $P$ is 3 cm directly above $B$. The length of $AB$ is 1 cm and the length of $BC$ is 1 cm. Find the angle that the triangle $ACP$ makes with the horizontal.




**7** **a** Solve $2\cos(2x + \pi) - 1 = 0$ for $-\pi \leq x \leq \pi$.

**b** Sketch the graph of $y = 2\cos(2x + \pi) - 1$ for $-\pi \leq x \leq \pi$, clearly labelling the axis intercepts.

**c** Solve $2\cos(2x + \pi) < 1$ for $-\pi \leq x \leq \pi$.

**8** The triangular base $ABC$ of a tetrahedron has side lengths $AB = 15$ cm, $BC = 12$ cm and $AC = 9$ cm. The apex $D$ is 9 cm vertically above $C$.

**a** Find the angle $C$ of the triangular base.

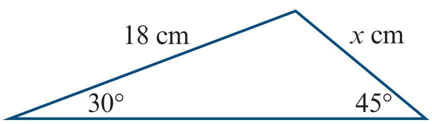
**b** Find the angles that the sloping edges make with the horizontal.

**9** Two ships sail from port at the same time. One sails 24 nautical miles due east in 3 hours, and the other sails 33 nautical miles on a bearing of 030° in the same time.

**a** How far apart are the ships 3 hours after leaving port?

**b** How far apart would they be in 5 hours if they maintained the same bearings and constant speed?

**10** Find $x$.




**11** An airport $A$ is 480 km due east of airport $B$. A pilot flies on a bearing of 225° from $A$ to $C$ and then on a bearing of 315° from $C$ to $B$.

**a** Make a sketch of the situation.

**b** Determine how far the pilot flies from $A$ to $C$.

**c** Determine the total distance the pilot flies.

**12** Find the equations of the asymptotes of the hyperbola with rule $x^2 - \dfrac{(y-2)^2}{9} = 15$.

**13** A curve is defined by the parametric equations $x = 3\cos(2t) + 4$ and $y = \sin(2t) - 6$. Give the Cartesian equation of the curve.

**14** A curve is defined by the parametric equations $x = 2\cos(\pi t)$ and $y = 2\sin(\pi t) + 2$. Give the Cartesian equation of the curve.

**15** **a** Sketch the graphs of $y = -2\cos x$ and $y = -2\cos\left(x - \dfrac{\pi}{4}\right)$ on the same set of axes, for $x \in [0, 2\pi]$.

**b** Solve $-2\cos\left(x - \dfrac{\pi}{4}\right) = 0$ for $x \in [0, 2\pi]$.

**c** Solve $-2\cos x < 0$ for $x \in [0, 2\pi]$.

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**16** Find all angles $\theta$ with $0 \le \theta \le 2\pi$, where:

**a** $\sin\theta = \dfrac{1}{2}$ **b** $\cos\theta = \dfrac{\sqrt{3}}{2}$ **c** $\tan\theta = 1$

**17** A circle has centre $(1, 2)$ and radius 3. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where $a$, $b$, $c$ and $d$ are positive constants, state the values of $a$, $b$, $c$ and $d$.

**18** Find the centre and radius of the circle with equation $x^2 + 8x + y^2 - 12y + 3 = 0$.

**19** Find the $x$- and $y$-axis intercepts of the ellipse with equation $\dfrac{x^2}{81} + \dfrac{y^2}{9} = 1$.

**20** The first term of an arithmetic sequence is $3p + 5$, where $p$ is a positive integer. The last term is $17p + 17$ and the common difference is 2.

**a** Find in terms of $p$:

**i** the number of terms **ii** the sum of the sequence.

**b** Show that the sum of the sequence is divisible by 14 only when $p$ is odd.

**21** A sequence is formed by using rising powers of 3 as follows: $3^0, 3^1, 3^2, \dots$.

**a** Find the $n$th term.

**b** Find the product of the first 20 terms.

**22** State the value of each of the following without using the absolute value function in your answer:

**a** $|-9|$ **b** $\left|-\dfrac{1}{400}\right|$ **c** $|9 - 5|$ **d** $|5 - 9|$ **e** $|\pi - 3|$ **f** $|\pi - 4|$

**23** **a** Let $f\colon \{x : |x| > 100\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$. State the range of $f$.

**b** Let $f\colon \{x : |x| < 0.1\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$. State the range of $f$.

**24** Let $f(x) = |x^2 - 3x|$. Solve the equation $f(x) = x$.

**25** For each of the following, sketch the graph of $y = f(x)$ and state the range of $f$:

**a** $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = 2|\sin x|$

**b** $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x^2 - 4x| - 3$

**c** $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3 - |x^2 - 4x|$

## Multiple-choice questions

**1** The 3rd term of a geometric sequence is 4 and the 8th term is 128. The 1st term is

**A** 2 **B** 1 **C** 32 **D** 5 **E** none of these

**2** If the numbers 5, $x$ and $y$ are in arithmetic sequence, then

**A** $y = x + 5$ **B** $y = x - 5$ **C** $y = 2x + 5$ **D** $y = 2x - 5$ **E** none of these

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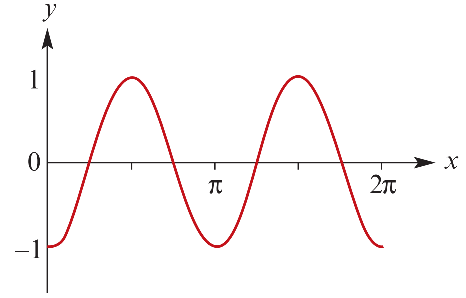
**3** If $2\cos x° - \sqrt{2} = 0$, then the value of the acute angle $x°$ is

**A** 30° **B** 60° **C** 45° **D** 25° **E** 27.5°

**4** The equation of the graph shown is




**A** $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

**B** $y = \cos\left(x + \frac{\pi}{4}\right)$

**C** $y = \sin(2x)$

**D** $y = -2\sin(x)$

**E** $y = \sin\left(x + \frac{\pi}{4}\right)$

**5** Which of the following recurrence relations generates the sequence 2, 6, 22, 86, 342, . . . ?

**A** $t_1 = 2,\ t_{n+1} = t_n + 4$ **B** $t_1 = 2,\ t_{n+1} = 2t_n + 2$ **C** $t_1 = 2,\ t_{n+1} = 3t_n$

**D** $t_1 = 2,\ t_{n+1} = 4t_n - 2$ **E** $t_1 = 2,\ t_{n+1} = 5t_n - 4$

**6** In a geometric sequence, $t_2 = 24$ and $t_4 = 54$. If the common ratio is positive, then the sum of the first five terms is

**A** 130 **B** 211 **C** 238 **D** 316.5 **E** 810

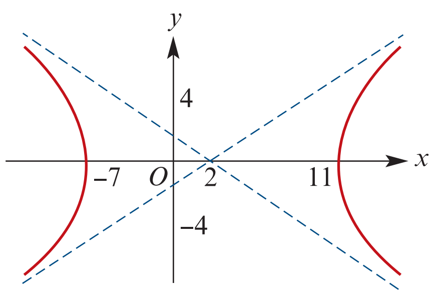
**7** In a triangle $ABC$, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of $c$ to the nearest whole number is

**A** 9 **B** 10 **C** 11 **D** 81 **E** 129

**8** The coordinates of the centre of the circle with equation $x^2 - 8x + y^2 - 2y = 8$ are

**A** $(-8, -2)$ **B** $(8, 2)$ **C** $(-4, -1)$ **D** $(4, 1)$ **E** $(1, 4)$

**9** The equation of the graph shown is




**A** $\frac{(x+2)^2}{27} - \frac{y^2}{108} = 1$

**B** $\frac{(x-2)^2}{9} - \frac{y^2}{34} = 1$

**C** $\frac{(x+2)^2}{81} - \frac{y^2}{324} = 1$

**D** $\frac{(x-2)^2}{81} - \frac{y^2}{324} = 1$

**E** $\frac{(x+2)^2}{9} - \frac{y^2}{36} = 1$

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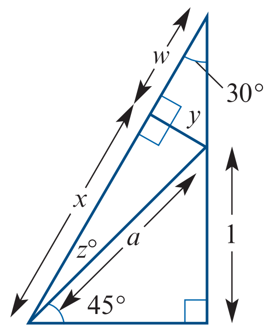
## Extended-response questions

**1** **a** Find the values of $a$, $y$, $z$, $w$ and $x$.

**b** Hence deduce exact values for $\sin 15°$, $\cos 15°$ and $\tan 15°$.

**c** Find the exact values of $\sin 75°$, $\cos 75°$ and $\tan 75°$.




**2** A hiker walks from point $A$ on a bearing of 010° for 5 km and then on a bearing of 075° for 7 km to reach point $B$.

**a** Find the length of $AB$.

**b** Find the bearing of $B$ from the start point $A$.

A second hiker walks from point $A$ on a bearing of 080° for 4 km to a point $P$, and then walks in a straight line to $B$.

**c** **i** Find the total distance travelled by the second hiker.

**ii** Find the bearing on which the hiker must travel in order to reach $B$ from $P$.

A third hiker also walks from point $A$ on a bearing of 080° and continues on that bearing until he reaches point $C$. He then turns and walks towards $B$. In doing so, the two legs of the journey are of equal length.

**d** Find the total distance travelled by the third hiker to reach $B$.

**3** An ellipse is defined by the rule $\dfrac{x^2}{2} + \dfrac{(y+3)^2}{5} = 1$.

**a** Find:

**i** the domain of the relation

**ii** the range of the relation

**iii** the centre of the ellipse.

An ellipse $E$ is given by the rule $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$. The domain of $E$ is $[-1, 3]$ and its range is $[-1, 5]$.

**b** Find the values of $a$, $b$, $h$ and $k$.

The line $y = x - 2$ intersects the ellipse $E$ at $A(1, -1)$ and at $P$.

**c** Find the coordinates of the point $P$.

A line perpendicular to the line $y = x - 2$ is drawn at $P$. This line intersects the $y$-axis at the point $Q$.

**d** Find the coordinates of $Q$.

**e** Find the equation of the circle through $A$, $P$ and $Q$.

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**4** **a** Show that the circle with equation $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the $x$-axis and the $y$-axis.

**b** Show that every circle that touches both the $x$-axis and the $y$-axis has an equation of a similar form.

**c** Hence show that there are exactly two circles that pass through the point $(2, 4)$ and just touch the $x$-axis and the $y$-axis, and give their equations.

**d** For each of these two circles, state the coordinates of the centre and give the radius.

**e** For each circle, find the gradient of the line which passes through the centre and the point $(2, 4)$.

**f** For each circle, find the equation of the tangent to the circle at the point $(2, 4)$.

**5** A circle is defined by the parametric equations $x = a\cos t$ and $y = a\sin t$. Let $P$ be the point with coordinates $(a\cos t, a\sin t)$.

**a** Find the equation of the straight line which passes through the origin and the point $P$.

**b** State the coordinates, in terms of $t$, of the other point of intersection of the circle with the straight line through the origin and $P$.

**c** Find the equation of the tangent to the circle at the point $P$.

**d** Find the coordinates of the points of intersection $A$ and $B$ of the tangent with the $x$-axis and the $y$-axis respectively.

**e** Find the area of triangle $OAB$ in terms of $t$ if $0 < t < \frac{\pi}{2}$. Find the value of $t$ for which the area of this triangle is a minimum.

**6** This diagram shows a straight track through points $A$, $S$ and $B$, where $A$ is 10 km northwest of $B$ and $S$ is exactly halfway between $A$ and $B$. A surveyor is required to reroute the track through $P$ from $A$ to $B$ to avoid a major subsidence at $S$. The surveyor determines that $A$ is on a bearing of 330° from $P$ and that $B$ is on a bearing of 070° from $P$. Assume the region under consideration is flat.

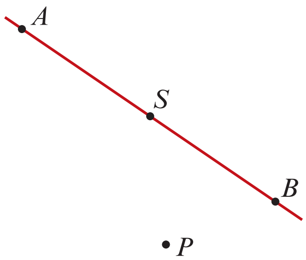
Find:

**a** the magnitudes of angles $APB$, $PAB$ and $PBA$

**b** the distance from $P$ to $B$ and from $P$ to $S$

**c** the bearing of $S$ from $P$

**d** the distance from $A$ to $B$ through $P$, if the surveyor chooses to reroute the track along a circular arc.




**7** Consider the function with rule $f(x) = |x^2 - ax|$, where $a$ is a positive constant.

**a** State the coordinates of the $x$-axis intercepts.

**b** State the coordinates of the $y$-axis intercept.

**c** Find the maximum value of the function in the interval $[0, a]$.

**d** Find the possible values of $a$ for which the point $(-1, 4)$ lies on the graph of $y = f(x)$.

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