2 Logic and proof

Objectives

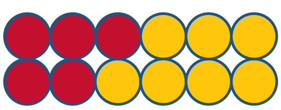
- To revise the concept of **divisibility** for integers.
- To revise basic concepts of **proof**, including:
  - conditional statements
  - equivalent statements
  - proof by contrapositive
  - proof by contradiction
  - counterexamples.
- To prove results involving **inequalities**.
- To evaluate **telescoping series**.
- To understand the **principle of mathematical induction**.
- To use mathematical induction to prove results involving:
  - divisibility
  - partial sums and products of sequences.

A **mathematical proof** is an argument that confirms the truth of a mathematical statement. It shows that a list of stated assumptions will guarantee a conclusion.

For example, consider the following simple claim involving even and odd numbers.

		Assumption		Conclusion
Claim	If	m and n are odd integers	then	$m + n$ is an even integer.

The truth of this claim is suggested by the picture on the right; an odd number of red dots can be combined with an odd number of yellow dots to give an even aggregate of dots.



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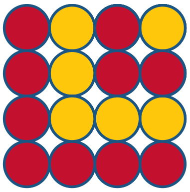
However, a more rigorous argument would proceed as follows. Assume that both m and n are odd integers. Then $m = 2a + 1$ and $n = 2b + 1$, for integers a and b. Therefore

$$\begin{aligned} m + n &= (2a + 1) + (2b + 1) \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1) \\ &= 2k \end{aligned}$$

where $k = a + b + 1$ is an integer. Hence $m + n$ is even.

The next claim has only one assumption.

		Assumption		Conclusion
Claim	If	n is a natural number	then	$1 + 3 + 5 + \cdots + (2n - 1) = n^2$.



This claim is illustrated by the picture on the right. Each L-shaped configuration of dots represents a different odd number. These can be nestled perfectly into the shape of a square.

So this picture gives great insight into why the claim is true. However, the picture alone does not constitute a proof. We need to show that the equation holds for every natural number n. We can prove results like this using *mathematical induction*.

In this chapter, we first revise concepts of logic and proof from Specialist Mathematics Units 1 & 2, before moving on to other applications of proof and mathematical induction.

Note: In the Interactive Textbook, each section of this chapter includes a skillsheet to provide further practice in areas such as sequences and series, combinatorics, matrices and graph theory.

2A Revision of proof techniques

We start by revising the fundamental ideas of proof introduced in Specialist Mathematics Units 1 & 2.

Divisibility of integers

- The set of **natural numbers** is $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.
- The set of **integers** is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Divisibility

Let a and b be integers. Then we say that a is **divisible** by b if there exists an integer k such that $a = kb$. In this case, we also say that b is a **divisor** of a.

Note: Alternatively, we can say that a is a **multiple** of b and that b is a **factor** of a.

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For example:

- 12 is divisible by 3, since $12 = 4 \times 3$
- -6 is divisible by 2, since $-6 = -3 \times 2$
- 0 is divisible by any integer n, since $0 = 0 \times n$.

On the other hand, the integer 14 is not divisible by 3. In this case, the best we can write is

$$14 = 4 \times 3 + 2$$

We say that 14 leaves a remainder of 2 when divided by 3. More generally, we have the following important result.

Euclidean division

If a and b are integers with $b \neq 0$, then there are unique integers q and r such that

$$a = qb + r \quad \text{where} \quad 0 \leq r < |b|$$

Note: Here q is the **quotient** and r is the **remainder** when a is divided by b.

Example 1

Let $n \in \mathbb{Z}$. Prove that $n^3 - n$ is divisible by 3.

Solution

Method 1

Note that $n^3 - n = n(n-1)(n+1)$.

When the integer n is divided by 3, the remainder must be 0, 1 or 2.
Therefore n can be written in the form $3k$, $3k+1$ or $3k+2$, for some integer k.

Case 1: $n = 3k$. Then
$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= 3k(3k-1)(3k+1) \end{aligned}$$

Case 2: $n = 3k+1$. Then
$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= (3k+1)(3k)(3k+2) \\ &= 3k(3k+1)(3k+2) \end{aligned}$$

Case 3: $n = 3k+2$. Then
$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= (3k+2)(3k+1)(3k+3) \\ &= 3(k+1)(3k+1)(3k+2) \end{aligned}$$

In all three cases, we see that $n^3 - n$ is divisible by 3.

Method 2

Note that $n^3 - n$ is the product of the three consecutive integers $n-1$, n and $n+1$.

In any set of three consecutive integers, one of the integers must be a multiple of 3. (This fact, although true, actually requires its own proof!) Therefore the product of three consecutive integers must also be a multiple of 3.

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Conditional statements and direct proof

The statement proved in Example 1 can be broken down into two parts:

Statement	If	n is an integer	then	$n^3 - n$ is divisible by 3.

This is an example of a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as $P \Rightarrow Q$, which is read as 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 2

Show that if n is an odd integer, then it is the sum of two consecutive integers.

Solution

Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. We can write

$$\begin{aligned} n &= 2k + 1 \\ &= k + (k + 1) \end{aligned}$$

Hence n is the sum of the consecutive integers k and $k + 1$.

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. Negation turns a true statement into a false statement, and a false statement into a true statement.

Statement	The sum of any two odd numbers is even.	(true)
Negation	There are two odd numbers whose sum is odd.	(false)

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

For example, we can use the second law to negate the following statement about integers a and b:

Statement	a is odd	or	b is odd
Negation	a is even	and	b is even

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Note: When negating a statement involving variables, it helps to know the set that each variable takes its value from. For example, if we know that a is an integer, then the negation of 'a is odd' is 'a is even'.

Proof by contrapositive

Consider this statement about an integer n:

Statement	If $n^2 + 2n$ is odd then n is odd.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If n is even then $n^2 + 2n$ is even.

Note that a conditional statement and its contrapositive are always logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 3

Let $n \in \mathbb{Z}$. Prove that if $n^2 + 2n$ is odd, then n is odd.

Solution

We will prove the contrapositive statement: If n is even, then $n^2 + 2n$ is even.

Assume that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 2n &= (2k)^2 + 2(2k) \\ &= 4k^2 + 4k \\ &= 2(2k^2 + 2k) \end{aligned}$$

Hence $n^2 + 2n$ is even, since $2k^2 + 2k \in \mathbb{Z}$.

As the contrapositive is equivalent to the original statement, we have proved the claim.

Proof by contradiction

The basic outline of a **proof by contradiction** is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

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Example 4

Suppose x satisfies $2^x = 5$. Using a proof by contradiction, show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 2^x = 5 \quad &\Rightarrow \quad 2^{\frac{m}{n}} = 5 \\ &\Rightarrow \quad \left(2^{\frac{m}{n}}\right)^n = 5^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 2^m = 5^n \end{aligned}$$

The left-hand side of this equation is even and the right-hand side is odd. This gives a contradiction, and so x is not rational.

The converse of a conditional statement

Consider this statement about integers m and n:

Statement	If	m and n are odd	then	$m + n$ is even.	(true)

By switching the hypothesis and the conclusion, we obtain the **converse** statement.

Converse	If	$m + n$ is even	then	m and n are odd.	(false)

The converse of a true statement may not be true.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Example 5

a Let n be an integer. Statement: If n^2 is divisible by 2, then n is divisible by 2.
Write the converse statement and show that it is true.

b Let S be a quadrilateral. Statement: If S is a square, then S has equal diagonals.
Write the converse statement and show that it is not true.

Solution

a Converse: If n is divisible by 2, then n^2 is divisible by 2.
Assume that n is divisible by 2. Then $n = 2k$ for some integer k. Therefore

$$n^2 = (2k)^2 = 2(2k^2)$$

which is divisible by 2.

b Converse: If S has equal diagonals, then S is a square.
The converse statement is false, since any rectangle has equal diagonals.

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Equivalent statements

Now consider the following two statements about a particular triangle:

P: The triangle has three equal sides.

Q: The triangle has three equal angles.

Both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements. We write $P \Leftrightarrow Q$.

We can also say that P is true **if and only if** Q is true. So in the above example, we can say that a triangle has three equal sides if and only if it has three equal angles.

To prove that two statements P and Q are equivalent, you have to prove two things:

$P \Rightarrow Q$ and $Q \Rightarrow P$

Example 6

Let $n \in \mathbb{Z}$. Prove that n is divisible by 3 if and only if n^2 is divisible by 3.

Solution

($\Rightarrow$) Assume that n is divisible by 3. We want to show that n^2 is divisible by 3.

Since n is divisible by 3, there exists an integer k such that $n = 3k$. Therefore $n^2 = (3k)^2 = 3(3k^2)$. Hence n^2 is divisible by 3.

($\Leftarrow$) Assume that n^2 is divisible by 3. We want to show that n is divisible by 3.

When the integer n is divided by 3, the remainder must be 0, 1 or 2. Therefore n can be written in the form $3k$, $3k + 1$ or $3k + 2$, for some integer k.

Case 1: $n = 3k$. This is the case where n is divisible by 3.

Case 2: $n = 3k + 1$. Then $n^2 = 9k^2 + 6k + 1$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Case 3: $n = 3k + 2$. Then $n^2 = 9k^2 + 12k + 4$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Hence n must be divisible by 3.

Skill-sheet

Exercise 2A

Divisibility of integers

1 Let n be an even integer. Prove that $n^2 + 2n$ is divisible by 4.

2 Let m and n be integers. Prove that $(2m + n)^2 - (2m - n)^2$ is divisible by 8.

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3 Assume that m is divisible by 3 and n is divisible by 5. Prove that:

a mn is divisible by 15

b m^2n is divisible by 45.

Example 1

4 **a** Let $n \in \mathbb{Z}$. Prove that $n^2 - n$ is even by considering the cases when n is odd and n is even.

b Provide another proof by factorising $n^2 - n$.

5 Consider integers m, n, a and b. Prove that if m is a divisor of a and n is a divisor of b, then mn is a divisor of ab.

Direct proof

Example 2

6 Show that if n is an odd integer, then $n^2 + 8n + 3$ is even.

7 Prove that if m and n are perfect cubes, then mn is a perfect cube.

8 **a** Factorise the expression $n^4 + 2n^3 - n^2 - 2n$.

b Use your factorised expression to provide a simple proof that $n^4 + 2n^3 - n^2 - 2n$ is divisible by 24 for all $n \in \mathbb{Z}$.

9 **a** Prove that if n is an odd integer, then there is an integer m such that $n^2 = 8m + 1$.

b Hence, prove that there is only one integer whose square has the form $2^k - 1$, where $k \in \mathbb{N}$.

10 Every integer n is of the form $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, for some integer k.

a Using this fact, prove that the cube of every integer n is of the form $9m$, $9m + 1$ or $9m + 8$, for some integer m.

b Explain why there are no cubes in the sequence $92, 992, 9992, 99992, \dots$.

11 Let $n \in \mathbb{Z}$. Prove that $3n^2 + 7n + 11$ is odd.
Hint: Consider the cases when n is odd and n is even.

12 Prove that, for any two positive integers that are not divisible by 3, the difference between their squares is divisible by 3.
Hint: If an integer n is not divisible by 3, then $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.

13 **a** Prove that every square number is of the form $5k$, $5k + 1$ or $5k + 4$, where $k \in \mathbb{Z}$.
Hint: Every natural number is of the form $5m$, $5m + 1$, $5m + 2$, $5m + 3$ or $5m + 4$, where $m \in \mathbb{Z}$.

b Hence, explain why no square number has a final digit equal to 2 or 3.

c Hence, determine how many square numbers appear in this list:

$$1!, \quad 1! + 2!, \quad 1! + 2! + 3!, \quad 1! + 2! + 3! + 4!, \quad \dots$$

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Proof by contrapositive

Example 3 **14** Let $a, b \in \mathbb{Z}$. Consider the statement: If ab is even, then a is even or b is even.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

15 Let $m, n \in \mathbb{Z}$. Consider the statement: If $m^2 + n^2$ is even, then $m + n$ is even.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

16 Let $n \in \mathbb{N}$. Consider the statement: If $8^n - 1$ is prime, then n is odd.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

c Is there anything special about the number 8 here? Can you generalise your proof?

17 Let $n \in \mathbb{Z}$. Prove that if n is even, then n cannot be expressed as the sum of two consecutive integers.

18 Let $x \in \mathbb{R}$. Prove that if x is irrational, then $2x - 3$ is irrational.

Proof by contradiction

Example 4 **19** Use proof by contradiction for each of the following:

a Prove that there is no largest natural number.

b Let $a, b \in \mathbb{R}$ such that $a + b > 100$. Show that $a > 50$ or $b > 50$.

c Let a and b be positive integers. Show that $a \le \sqrt{ab}$ or $b \le \sqrt{ab}$.

d Prove that $\log_2 7$ is irrational.

e Let $a, b \in \mathbb{R}$ such that a is rational and b is irrational. Show that $a + b$ is irrational.

f Prove that the product of two consecutive natural numbers is never a square number.

g Let $n \in \mathbb{N}$ and assume that n, $n + 2$ and $n + 4$ are all prime. Show that $n = 3$.

20 Define the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x}{x-1}$. Prove, by way of contradiction, that 1 does not belong to the range of f.

21 Define the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x^2}{x-1}$. Prove, by way of contradiction, that 1 does not belong to the range of f.

The converse of a conditional statement

Example 5 **22** Let m and n be integers. For each of the following statements, write down the converse statement. Decide whether the converse is true or false, and explain why.

a If $3n$ is odd, then n is odd.

b If m is even and n is odd, then mn is even.

c If n is divisible by 6, then n is divisible by 2 and 3.

d If n is divisible by 24, then n is divisible by 4 and 6.

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Equivalent statements

23 Let n be an integer. Prove that n is even if and only if $n + 1$ is odd.

24 Let a, m and n be integers, where $m \neq 0$. Prove that nm is a divisor of am if and only if n is a divisor of a.

Example 6

25 **a** Let $n \in \mathbb{N}$. Prove that n is divisible by 2 if and only if n^2 is divisible by 2.

b Hence, prove that $\sqrt{2q}$ is irrational, whenever q is an odd natural number.

c Using this fact, prove that $\sqrt{2} + \sqrt{3}$ is irrational.

26 Let n be an integer. Prove that n is divisible by 3 if and only if n^3 is divisible by 9.

27 **a** Write the number 99 as the sum of three consecutive integers.

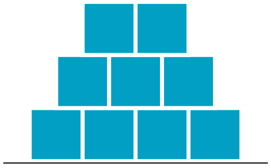
b Let n be an integer. Prove that n is divisible by 3 if and only if n can be written as the sum of three consecutive integers.

Mixed proof questions

28 Prove that the sum of three consecutive positive integers is a divisor of the sum of their cubes.

29 Let $k \in \mathbb{N}$. Prove that the product of k consecutive positive integers is divisible by $k!$.
Hint: Consider the binomial coefficient ${}^{n+k}C_k$.

30 We will say that a natural number n is **stackable** if it is possible to form a tower of n blocks with at least two rows in such a way that every row above the bottom row has exactly one less block than the row below. For example, the number 9 is stackable, as shown in the diagram.



Prove that no power of 2 is stackable. Hint: Use a proof by contradiction.

31 Notice that

$$1 = 1^2, \qquad 2 = -1^2 - 2^2 - 3^2 + 4^2, \qquad 3 = -1^2 + 2^2$$

a Let m be an integer. Prove that

$$(m+1)^2 - (m+2)^2 - (m+3)^2 + (m+4)^2 = 4$$

b Hence, prove that every natural number can be written in the form

$$\pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$$

for some value of n and a suitable choice of sign for each term.

32 In this question, we will give a proof that there are infinitely many prime numbers.

a Let $m \in \mathbb{N}$. Prove that if d is a divisor of both m and $m + 1$, then $d = 1$.

b Now let $n \in \mathbb{N}$ and assume that p is a prime factor of $n! + 1$. Prove that $p > n$.
Hint: Suppose that $p \leq n$.

c Why does this mean that there are infinitely many primes?

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2B Quantifiers and counterexamples

Quantification using 'for all' and 'there exists'

For all

A **universal statement** claims that a property holds for *all* members of a given set. Such a statement can be written using the quantifier '**for all**'. For example:

Statement	For all real numbers x and y, we have $x^2 + 5y^2 \geq 2xy$.

To prove that this statement is true, we need to give a general argument that applies for every choice of real numbers x and y. We will prove inequalities like this in the next section.

There exists

An **existence statement** claims that a property holds for *at least one* member of a given set. Such a statement can be written using the quantifier '**there exists**'. For example:

Statement	There exists a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$.

To prove that this statement is true, we just need to give one instance. The triple $(3, 4, 5)$ provides an example, since $3^2 + 4^2 = 5^2$.

Example 7

Rewrite each statement using either 'for all' or 'there exists':

a Some real numbers are irrational.

b Every integer that is divisible by 4 is also divisible by 2.

Solution

a There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

b For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negation without quantifiers

We discussed negation in the previous section, and we used De Morgan's laws to negate statements involving 'and' and 'or'. It is also helpful to be able to negate statements involving 'implies'.

Consider the conditional statement 'If you study Mathematics, then you study Physics'. The only way this can be false is if you are studying Mathematics but not Physics. So the negation of the statement is 'You study Mathematics and you do not study Physics'.

Negations of basic compound statements

- not (P and Q) is equivalent to (not P) or (not Q)
- not (P or Q) is equivalent to (not P) and (not Q)
- not ($P \Rightarrow Q$) is equivalent to P and (not Q)

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Negation with quantifiers

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

Example 8

Write down the negation of each of the following statements:

a For all natural numbers n, we have $2n \geq n + 1$.

b There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

c For all real numbers x and y, if $x < y$, then $x^2 < y^2$.

Solution

a There exists a natural number n such that $2n < n + 1$.

b For all integers m, we have $m^2 \neq 4$ or $m^3 \neq -8$.

c There exist real numbers x and y such that $x < y$ and $x^2 \geq y^2$.

Notation for quantifiers

The words 'for all' can be abbreviated using the *turned A* symbol, ∀. The words 'there exists' can be abbreviated using the *turned E* symbol, ∃. For example, the two statements considered at the start of this section can be written in symbols as follows:

- $(\forall x, y \in \mathbb{R})\ \ x^2 + 5y^2 \geq 2xy$
- $(\exists (a, b, c) \in \mathbb{Z}^3)\ \ a^2 + b^2 = c^2$

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

Disproving universal statements

We have seen that a universal statement claims that a property holds for *all* members of a given set. For example:

Statement	For all real numbers x, the number $x^2 - x$ is positive.

So to disprove a universal statement, we simply need to give one example where the property does not hold. Such an example is called a **counterexample**.

Example 9

Disprove the statement: For all real numbers x, the number $x^2 - x$ is positive.

Solution

When $x = 0$, we obtain $x^2 - x = 0$, which is not positive.

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Example 10

A function $f: \mathbb{R} \to \mathbb{R}$ is **strictly increasing** if $a < b$ implies $f(a) < f(b)$, for all $a, b \in \mathbb{R}$.

Disprove the statement: If a function $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing and differentiable, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

Solution

The statement is not true, as the function $f(x) = x^3$ is a counterexample. The function f is strictly increasing and differentiable, but $f'(0) = 0$.

Note that the negation of a universal statement is an existence statement. For example:

Statement	For all $x, y \in \mathbb{R}$, if $x < y$, then $x^2 < y^2$.
Negation	There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 \geq y^2$.

Clearly, the universal statement above is false, because there exist real numbers for which the property does not hold. For example: $-1 < 0$ but $(-1)^2 \geq 0^2$.

Disproving existence statements

Consider this existence statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 7n + 12$ is a prime number.

To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 7n + 12$ is not a prime number.

The negation is easy to prove, since

$$n^2 + 7n + 12 = (n + 3)(n + 4)$$

is clearly a composite number for each $n \in \mathbb{N}$. As this example demonstrates, the negation of an existence statement is a universal statement.

Example 11

Disprove each of the following statements:

a There exists $n \in \mathbb{N}$ such that $n^2 + 15n + 56$ is a prime number.

b There exists some real number x such that $x^2 = -1$.

Solution

a We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 15n + 56$ is not prime. This is true, since

$$n^2 + 15n + 56 = (n + 7)(n + 8)$$

is a composite number for each $n \in \mathbb{N}$.

b We need to prove that, for all real numbers x, we have $x^2 \neq -1$. This is true, since for every real number x, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

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Skill-sheet

Exercise 2B

Example 7

1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

a For each $n \in \mathbb{N}$, the number $(2n + 1)^2$ is odd.
b There is an even prime number.
c For every integer n, the integer $n(n + 1)$ is even.
d All squares have four sides.
e Some natural numbers are composites.
f At least one real number x satisfies the equation $x^2 - 2x - 5 = 0$.
g Any real number has a cube root.
h The angle sum of a quadrilateral is 360°.

Example 8

2 Write down the negation of each of the following statements:

a For all $x \in \mathbb{R}$, we have $x^2 \geq 0$.
b For every natural number n, the number $n^2 + n + 11$ is prime.
c There exist prime numbers p and q for which $p + q = 100$.
d For all $x \in \mathbb{R}$, if $x > 0$, then $x^3 > x$.
e There exist integers a, b and c such that $a^3 + b^3 = c^3$.
f For all $x, y \in \mathbb{R}$, we have $(x + y)^3 = x^3 + y^3$.
g There exist $x, y \in \mathbb{R}$ such that $x \geq y$ and $x^2 \leq y^2$.
h There exists a real number x such that $x^2 + x + 1 = 0$.
i For all natural numbers n, if n is not divisible by 3, then $n^2 + 2$ is divisible by 3.
j For every integer m, if $m > 2$ or $m < -2$, then $m^2 > 4$.
k For all integers m and n, we have that mn is even or $m + n$ is even.
l There exists a rational number a for which $\sqrt{2} \cdot a$ is rational.
m For all real numbers x, if $x \in (-1, 1)$, then $x^2 < 1$.
n There exist real numbers x and y such that $xy > 0$ and $x + y < 0$.

Example 9
Example 10

3 Provide a counterexample for each of the following statements:

a For all natural numbers n, the number $n^2 + n + 1$ is prime.
b For all real numbers x, we have $x^2 > 0$.
c For all $a, b \in \mathbb{R}$, if a and b are irrational, then $a + b$ is irrational.
d For all $a, b \in \mathbb{R}$, if a and b are irrational, then ab is irrational.
e For all real numbers a, b and c, if $ab = ac$, then $b = c$.
f For each $n \in \mathbb{N}$, if n^2 is divisible by 4, then n is divisible by 4.
g For all $a, b, c \in \mathbb{Z}$, if c is a divisor of ab, then c is a divisor of a or c is a divisor of b.
h For all integers m and n, if n^2 is a divisor of m^3, then n is a divisor of m.
i If a function $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing, then its graph crosses the x-axis.
j If f is a differentiable function and $f'(0) = 0$, then f has a turning point at $(0, f(0))$.
k If every vertex of a graph has degree at least 1, then the graph is connected.

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4 The 2×2 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the 2×2 zero matrix is $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Provide a counterexample for each of the following statements:

a For all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{AB} = \mathbf{BA}$.

b For each 2×2 matrix $\mathbf{A}$, if $\mathbf{A}^2 = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$.

c For each 2×2 matrix $\mathbf{A}$, if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} = \mathbf{I}$.

Example 11

5 Show that each of the following existence statements is false:

a There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.

b There exists $n \in \mathbb{N}$ such that $n^2 + 11n + 30$ is a prime number.

c There exists $x \in \mathbb{R}$ such that $5 + 2x^2 = 1 + x^2$.

2C Proving inequalities

An **inequality** is a statement that orders two real numbers.

$x < y$	x is less than y
$x \leq y$	x is less than or equal to y
$x > y$	x is greater than y
$x \geq y$	x is greater than or equal to y

Possibly the most important inequality is the statement that $x^2 \geq 0$ for every real number x. This simply says that the square of any real number is non-negative. It important because so many inequalities depend on this result.

Example 12

Let x and y be real numbers. Prove that $x^2 + 5y^2 \geq 2xy$.

Solution

We will prove this result by showing that

$$x^2 - 2xy + 5y^2 \geq 0$$

To show this, we can complete the square (thinking of x as the variable and y as a constant). We find that

$$\begin{aligned} x^2 - 2xy + 5y^2 &= (x^2 - 2xy + y^2) - y^2 + 5y^2 \\ &= (x - y)^2 + 4y^2 \\ &= (x - y)^2 + (2y)^2 \\ &\geq 0 \end{aligned}$$

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Example 13

Prove that the area of a rectangle is no more than the area of a square with the same perimeter.

Solution

Let x and y be the side lengths of a rectangle. Its perimeter is $2x + 2y$.

The side length of a square with the same perimeter is $\frac{1}{4}(2x + 2y) = \frac{x+y}{2}$.

Therefore

$$\begin{aligned} \text{Area of square} - \text{Area of rectangle} &= \left(\frac{x+y}{2}\right)^2 - xy \\ &= \frac{x^2 + 2xy + y^2}{4} - \frac{4xy}{4} \\ &= \frac{x^2 - 2xy + y^2}{4} \\ &= \left(\frac{x-y}{2}\right)^2 \geq 0 \end{aligned}$$

In the above example, we see a proof of the following useful inequality.

AM–GM inequality

For $x, y \geq 0$, the arithmetic mean is greater than or equal to the geometric mean:

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Note: The two means are equal if and only if $x = y$.

We can use this inequality to give quick proofs of many results.

Example 14

a Suppose $a, b > 0$ and $ab = 24$. Using the AM–GM inequality, prove that $2a + 3b \geq 24$.

b Suppose $a, b \geq 0$ and $2a + 3b = 12$. Using the AM–GM inequality, prove that $ab \leq 6$.

Solution

a Assume $ab = 24$.
Using the AM–GM inequality
(with $x = 4a$ and $y = 6b$) we find

$$\begin{aligned} 2a + 3b &= \frac{4a + 6b}{2} \\ &\geq \sqrt{(4a)(6b)} \\ &= \sqrt{24ab} \\ &= \sqrt{24 \times 24} \\ &= 24 \end{aligned}$$

b Assume $2a + 3b = 12$.
Using the AM–GM inequality
(with $x = 2a$ and $y = 3b$) we find

$$\begin{aligned} ab &= \frac{1}{6}(2a)(3b) \\ &\leq \frac{1}{6}\left(\frac{2a + 3b}{2}\right)^2 \\ &= \frac{1}{6}\left(\frac{12}{2}\right)^2 \\ &= 6 \end{aligned}$$

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Skill-sheet

Exercise 2C

Example 12

1 Let $b \geq a \geq 0$. Prove that $\dfrac{b}{b+1} \geq \dfrac{a}{a+1}$.

2 Given that a and b are positive real numbers, prove that $a^3 + b^3 \geq a^2b + ab^2$.

3 **a** Prove that $11\sqrt{10} \geq 10\sqrt{11}$.

b We can prove a much more general statement. Let $a \geq b \geq 0$. Prove that $a\sqrt{b} \geq b\sqrt{a}$.

4 Let $a > 0$. Prove that $a + \dfrac{1}{a} \geq 2$.

5 **a** Prove that $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ when $x, y \in \mathbb{R}^+$.

b Prove that $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)(x + y) \geq 4$ when $x, y \in \mathbb{R}^+$.

c Prove that $\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)(x + y + z) \geq 9$ when $x, y, z \in \mathbb{R}^+$.

Example 13

6 **a** Let $x, y \geq 0$. Prove that

$$\left(\frac{x+y}{2}\right)^2 \leq \frac{x^2 + y^2}{2}$$

b Two pieces of string are used to form two squares, with sides of length x and y respectively. These two pieces of string are joined together and then cut in half. The two new pieces of string are used to form two squares of equal size. Prove that the total area has not increased.

Example 14

7 Let $a, b, c \geq 0$. Use the AM–GM inequality to prove each of the following:

a If $ab = 9$, then $a + b \geq 6$.

b If $a + b = 4$, then $ab \leq 4$.

c If $ab = 48$, then $3a + 4b \geq 24$.

d If $3a + 4b = 24$, then $ab \leq 12$.

e If $a + b + c = 1$, then $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq 1$.

8 Use the AM–GM inequality to prove that $(a + b)(b + c)(c + a) \geq 8abc$ for all $a, b, c \geq 0$.

9 **a** Assume $0 < a < 1$. Prove that $a > a^2$.

b Let θ be an acute angle. Using part **a**, prove that

$$\cos\theta + \sin\theta > 1$$

c Prove that

$$\cos\theta + \sin\theta \leq \sqrt{2}$$

10 Let a, b and c be real numbers.

a Prove that $a^2 + b^2 \geq 2ab$.

b Hence, prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c Hence, prove that $3(a^4 + b^4 + c^4) \geq (a^2 + b^2 + c^2)^2$.

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2D Telescoping series

The technique demonstrated in this section is called **telescopic cancelling**. In the solution of the following example, you will notice that a sum with $2n$ terms cancels down to a sum with just two terms. The sum collapses in a similar manner to a traditional extendable telescope.

This technique can be used to find the partial sums of some sequences. In the next section, we will consider another approach to proving some of these results: mathematical induction.

Example 15

a Find constants a and b such that $\dfrac{1}{k(k+1)} = \dfrac{a}{k} + \dfrac{b}{k+1}$ for all $k \in \mathbb{N}$.

b Hence, prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Solution

a We aim to find the partial fraction decomposition of the left-hand side. We have

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

$$\therefore \quad \frac{1}{k(k+1)} = \frac{a(k+1)+bk}{k(k+1)}$$

$$\therefore \quad 1 = a(k+1) + bk$$

$$\therefore \quad 1 = a + (a+b)k$$

Equating coefficients, we find that $a = 1$ and $a + b = 0$. Therefore $b = -1$. We have found the partial fraction decomposition to be

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

b We use the result from part **a** to expand each term of the series:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \qquad \text{(regrouping)}$$

$$= 1 - \frac{1}{n+1} \qquad \text{(cancelling)}$$

$$= \frac{n}{n+1}$$

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Skill-sheet

Exercise 2D

Example 15

1 **a** Using partial fractions, find real numbers a and b such that

$$\frac{1}{k(k+2)} = \frac{a}{k} + \frac{b}{k+2}$$

b Hence, evaluate each of the following sums:

i $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{97 \cdot 99}$ **ii** $\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{98 \cdot 100}$

2 Show that

$$\log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{99}{100}\right) = -2$$

3 **a** Show that $m \cdot m! = (m+1)! - m!$.

b Hence, prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

4 **a** Show that $\frac{m}{(m+1)!} = \frac{1}{m!} - \frac{1}{(m+1)!}$.

b Hence, prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

5 **a** Show that

$$k(k+1) = \frac{1}{3}\big(k(k+1)(k+2) - (k-1)k(k+1)\big)$$

b Let $n \in \mathbb{N}$. Use part **a** to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

6 **a** Using partial fractions, find real numbers a, b and c such that

$$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}$$

b Hence, prove that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

7 **a** Show that

$$\frac{\log_{10}\left(\frac{a}{b}\right)}{\log_{10}(a)\,\log_{10}(b)} = \frac{1}{\log_{10}(b)} - \frac{1}{\log_{10}(a)}$$

b Hence, evaluate

$$\frac{\log_{10}\left(\frac{2}{3}\right)}{\log_{10}(2)\,\log_{10}(3)} + \frac{\log_{10}\left(\frac{3}{4}\right)}{\log_{10}(3)\,\log_{10}(4)} + \cdots + \frac{\log_{10}\left(\frac{19}{20}\right)}{\log_{10}(19)\,\log_{10}(20)}$$

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2E Mathematical induction

In Example 15 from the previous section, we obtained the result

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

This result involves an infinite sequence of propositions, one for each natural number n:

$$P(1):\quad \frac{1}{1\cdot 2}=\frac{1}{1+1}$$

$$P(2):\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{2+1}$$

$$P(3):\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{3+1}$$

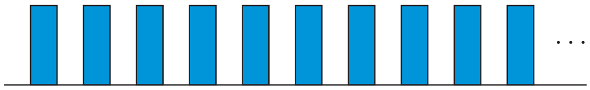
$$\vdots$$

We proved that the proposition $P(n)$ is true for every natural number n. In this section, we give an alternative proof of this result using **mathematical induction**.

The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

1. The first domino is knocked over.
2. Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.

b Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1)\text{ is true} \quad\Rightarrow\quad P(2)\text{ is true} \quad\Rightarrow\quad P(3)\text{ is true} \quad\Rightarrow\quad \cdots$$

Let's see how mathematical induction is used in practice.

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Using induction for partial sums

Mathematical induction is useful for proving many results about partial sums.

Example 16

Using mathematical induction, prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Step 1 $P(1)$ is the proposition $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$, that is, $\frac{1}{2} = \frac{1}{2}$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{(using } P(k)\text{)} \\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

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Using induction for divisibility results

We now use mathematical induction to prove results about divisibility. You should compare the next example with Example 1 from the start of this chapter.

Example 17

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$n^3 - n$ is divisible by 3

Step 1 $P(1)$ is the proposition $1^3 - 1 = 0$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$k^3 - k = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k + 1)$ is true, that is, we have to prove that the number $(k + 1)^3 - (k + 1)$ is divisible by 3. We have

$$\begin{aligned}(k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3m + 3k^2 + 3k && \text{(using } P(k)\text{)} \\ &= 3(m + k^2 + k)\end{aligned}$$

Therefore $(k + 1)^3 - (k + 1)$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Example 18

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. So $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

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Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3.

We have

$$\begin{aligned}
7^{k+1} - 4 &= 7 \cdot 7^k - 4 \\
&= 7(3m+4) - 4 \qquad \text{(using } P(k)\text{)} \\
&= 21m + 28 - 4 \\
&= 21m + 24 \\
&= 3(7m+8)
\end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Notation for sums

The sum of the first n squares can be written in two different ways:

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$$

The notation on the right-hand side is called **sigma notation**, and is a convenient shorthand for the **expanded form** you see on the left-hand side. The notation uses the symbol Σ, which is the uppercase Greek letter *sigma*. This is the Greek equivalent to the Roman letter S, with S here standing for the word *sum*.

More generally, if m and n are integers with $m \le n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This is read as 'the sum of the numbers a_i from $i = m$ to $i = n$'.

Example 19

Write $\sum_{i=1}^{5} 2^i$ in expanded form and evaluate.

Solution

$$\begin{aligned}
\sum_{i=1}^{5} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\
&= 2 + 4 + 8 + 16 + 32 \\
&= 62
\end{aligned}$$

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You may prefer to use this notation in induction proofs involving partial sums. The next example uses this notation to give the sum of the cubes of the first n odd numbers.

Example 20

Prove using the principle of mathematical induction that

$$\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2-1)$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2-1)$$

Step 1 First consider $P(1)$. Let $n = 1$ so that

$$\text{LHS of } P(1) = \sum_{i=1}^{1} (2i-1)^3 = (2 \cdot 1 - 1)^3 = 1$$

$$\text{RHS of } P(1) = 1^2(2 \cdot 1^2 - 1) = 1$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\sum_{i=1}^{k} (2i-1)^3 = k^2(2k^2-1)$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\sum_{i=1}^{k+1} (2i-1)^3 = (k+1)^2(2(k+1)^2-1)$$

We have

$$\begin{aligned}\text{LHS of } P(k+1) &= \sum_{i=1}^{k+1} (2i-1)^3 \\ &= \sum_{i=1}^{k} (2i-1)^3 + (2(k+1)-1)^3 \\ &= k^2(2k^2-1) + (2k+1)^3 && \text{(using } P(k)) \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1\end{aligned}$$

$$\begin{aligned}\text{RHS of } P(k+1) &= (k+1)^2(2(k+1)^2-1) \\ &= 2(k+1)^4 - (k+1)^2 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1\end{aligned}$$

Therefore $P(k+1)$ is true. Hence we have shown that $P(k)$ implies $P(k+1)$, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

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Notation for products

As with finite sums of numbers, we also have an efficient shorthand for expressing finite products of numbers.

For example, the product of the first n odd numbers can be written as

$$1 \times 3 \times 5 \times \cdots \times (2n-1) = \prod_{i=1}^{n} (2i-1)$$

The notation on the right-hand side is called **pi notation**, and is a convenient shorthand for the **expanded form** you see on the left-hand side. The notation uses the symbol Π, which is the uppercase Greek letter *pi*. This is the Greek equivalent to the Roman letter P, with P here standing for the word *product*.

More generally, if m and n are integers with $m \le n$, then

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times a_{m+2} \times \cdots \times a_n$$

This is read as 'the product of the numbers a_i from $i = m$ to $i = n$'.

Example 21

Write each of the following in expanded form and evaluate:

a $\displaystyle\prod_{i=1}^{3} i$ **b** $\displaystyle\prod_{i=1}^{5} (2i-1)$

Solution

a $\displaystyle\prod_{i=1}^{3} i = 1 \times 2 \times 3 = 6$

b
$$\begin{aligned}\prod_{i=1}^{5} (2i-1) &= (2 \cdot 1 - 1) \times (2 \cdot 2 - 1) \times (2 \cdot 3 - 1) \times (2 \cdot 4 - 1) \times (2 \cdot 5 - 1) \\ &= 1 \times 3 \times 5 \times 7 \times 9 \\ &= 945\end{aligned}$$

You may prefer to use this notation in induction proofs involving partial products.

In the next example, we give a proof by induction that

$$\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \cdots \times \left(1 + \frac{1}{n}\right) = n + 1$$

Using pi notation, this is expressed more compactly as

$$\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = n + 1$$

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Example 22

Using the principle of mathematical induction, prove that

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$$

Step 1 First consider $P(1)$. Let $n = 1$ so that

$$\text{LHS of } P(1)=\prod_{i=1}^{1}\left(1+\frac{1}{i}\right)=1+\frac{1}{1}=2$$

$$\text{RHS of } P(1)=1+1=2$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)=k+1$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=(k+1)+1$$

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= \prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)\\
&= \left(1+\frac{1}{k+1}\right)\times\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\\
&= \left(1+\frac{1}{k+1}\right)(k+1) && \text{(using } P(k)\text{)}\\
&= k+1+\frac{k+1}{k+1}\\
&= (k+1)+1\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true. Hence we have shown that $P(k)$ implies $P(k+1)$, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

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Using induction to prove inequalities

Mathematical induction can also be used to prove various inequalities. For most induction proofs, the base case is $n = 1$. In the next example, the base case is $n = 4$.

Example 23

Prove by induction that $3^n > n^3$ for all natural numbers $n \geq 4$.

Solution

For each natural number $n \geq 4$, let $P(n)$ be the proposition:

$$3^n > n^3$$

Step 1 First consider $P(4)$. Let $n = 4$ so that

$$\text{LHS of } P(4) = 3^4 = 81$$
$$\text{RHS of } P(4) = 4^3 = 64$$

Therefore $P(4)$ is true.

Step 2 Let k be a natural number with $k \geq 4$, and assume $P(k)$ is true. That is,

$$3^k > k^3$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$3^{k+1} > (k + 1)^3$$

Note that the right-hand side of $P(k + 1)$ expands to $k^3 + 3k^2 + 3k + 1$. This is what we are aiming for in the calculation below.

We have

$$\begin{aligned}
\text{LHS of } P(k + 1) &= 3^{k+1} \\
&= 3 \cdot 3^k \\
&> 3 \cdot k^3 && \text{(using } P(k)\text{)} \\
&= k^3 + k^3 + k^3 \\
&> k^3 + 3k^2 + 3k^2 && \text{(since } k \geq 4\text{)} \\
&= k^3 + 3k^2 + k^2 + k^2 + k^2 \\
&> k^3 + 3k^2 + 3k + 1 && \text{(since } k \geq 4\text{)} \\
&= (k + 1)^3 \\
&= \text{RHS of } P(k + 1)
\end{aligned}$$

Therefore $P(k + 1)$ is true. Hence we have shown that $P(k)$ implies $P(k + 1)$, for each $k \in \mathbb{N}$ with $k \geq 4$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers $n \geq 4$.

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Skill-sheet

Exercise 2E

Example 16

1 Prove each of the following using mathematical induction:

a $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1 + 3 + 5 + \cdots + (2n-1) = n^2$

c $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

e $1 + x + x^2 + \cdots + x^n = \dfrac{1-x^{n+1}}{1-x}$, where $x \neq 1$

f $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

g $2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$

h $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

Example 17
Example 18

2 Prove that each of the following divisibility statements is true for all $n \in \mathbb{N}$:

a $11^n - 1$ is divisible by 10
b $7^n - 3^n$ is divisible by 4
c $8^n - (-6)^n$ is divisible by 14
d $2^{3n+1} + 5$ is divisible by 7
e $4^{2n+1} + 5^{2n+1}$ is divisible by 9

3 **a** Prove by induction that $2^{2n} - 1$ is a multiple of 3 for all $n \in \mathbb{N}$.

b Provide a different proof of this result by first writing $2^{2n} - 1$ as the difference of two squares.

Example 19

4 Write each of the following in expanded form and evaluate:

a $\displaystyle\sum_{i=1}^{4} i^3$
b $\displaystyle\sum_{k=1}^{5} 3^k$
c $\displaystyle\sum_{i=0}^{3} (-1)^i\, i$
d $\displaystyle\frac{1}{5}\sum_{i=1}^{5} i$

e $\displaystyle\sum_{i=1}^{4} 2i$
f $\displaystyle\sum_{k=1}^{4} (k-1)^2$
g $\displaystyle\sum_{i=1}^{3} (i-2)^2$
h $\displaystyle\sum_{i=1}^{4} (2i-1)^2$

i $\displaystyle\sum_{i=1}^{3} r^i$
j $\displaystyle\sum_{i=1}^{3} i \cdot 2^i$
k $\displaystyle\sum_{i=0}^{3} 3^{3-i}$
l $\displaystyle\sum_{i=1}^{2} (x-1)^i$

m $\displaystyle\sum_{i=3}^{3} i^2$
n $\displaystyle\sum_{i=-2}^{2} i$
o $\displaystyle\sum_{i=1}^{n} 1$
p $\displaystyle\sum_{i=-4}^{-2} 2i$

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Example 20

5 Prove each of the following using mathematical induction:

a $\displaystyle\sum_{m=1}^{n}(2m-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ **b** $\displaystyle\sum_{m=1}^{n}(-1)^{m+1}m^2 = (-1)^{n+1}\frac{n(n+1)}{2}$

Example 21

6 Write each of the following in expanded form and evaluate:

a $\displaystyle\prod_{i=1}^{4} i$ **b** $\displaystyle\prod_{j=1}^{3} 2j$ **c** $\displaystyle\prod_{k=1}^{3} k^2$ **d** $\displaystyle\prod_{i=0}^{2} 10^i$

e $\displaystyle\prod_{j=1}^{4} \frac{j}{j+1}$ **f** $\displaystyle\prod_{k=1}^{5} \sqrt{k}$ **g** $\displaystyle\prod_{i=1}^{3} \left(\frac{i+1}{i}\right)^2$ **h** $\displaystyle\prod_{i=0}^{4} \frac{1}{2^i}$

Example 22

7 Using mathematical induction, prove that

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right) = \frac{n+1}{2n}$$

for all natural numbers $n \geq 2$.

8 **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

b Find a nicer proof involving factorisation that works for all $n \in \mathbb{Z}$.

Example 23

9 Use induction to prove each of the following. (Note that the base case is not $n = 1$.)

a $2^n > n^2$ for all $n \geq 5$ **b** $n! > 2^n$ for all $n \geq 4$

c $4^n > 2 \times 3^n$ for all $n \geq 3$ **d** $3^n > 2n + 1$ for all $n \geq 2$

10 **a** Prove, by mathematical induction, that $n^3 + 3n^2 + 2n$ is divisible by 6 for all $n \in \mathbb{N}$.

b Prove the result without mathematical induction by instead factorising the expression.

11 **a** Show that $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (bx - ay)^2$.

b Write 13 as the sum of two squares.

c Hence, using mathematical induction, prove that 13^n can be written as the sum of two squares, for every natural number n.

12 Let $m \in \mathbb{N}$. Prove by induction that if m is odd, then m^n is odd, for every $n \in \mathbb{N}$.

13 The Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$.

a Find f_n for $n = 1, 2, \dots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to find a general formula for the expression from part **c**.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that f_{5n} is divisible by 5 for all $n \in \mathbb{N}$.

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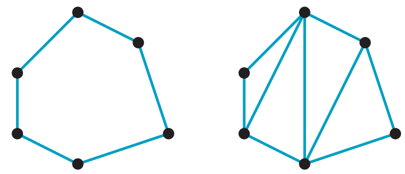
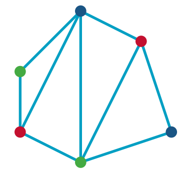
14 A polygonal area can be **triangulated** if we can add extra edges between the vertices so that the polygon is a union of non-intersecting triangles.



a Using mathematical induction, prove that for all $n \geq 3$, every convex polygonal area with n vertices can be triangulated.

b Consider a triangulation of a convex polygon with n vertices, where $n \geq 3$. Using mathematical induction, prove that we can colour the vertices using three colours, in such a way that no two adjacent vertices have the same colour.



15 Using induction, prove that

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \geq \sqrt{n}$$

for every natural number n.

16 The Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove that, for each $n \geq 2$, the following matrix equation holds:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

17 **a** Prove by induction that $3^n - (-2)^n$ is divisible by 5 for all $n \in \mathbb{N}$.

b Prove by induction that $4^n - (-3)^n$ is divisible by 7 for all $n \in \mathbb{N}$.

c Generalise the previous two results. Prove your generalisation using induction.

18 Prove each of the following using mathematical induction:

a $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

b $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2$

19 Let $f \colon [0,1] \to \mathbb{R}$, $f(x) = \dfrac{x}{2-x}$.

a Determine the rule for $(f \circ f)(x) = f(f(x))$, the composition of f with itself.

b For $x \in [0,1]$ and $n \in \mathbb{N}$, let

$$F_n(x) = (\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies of } f})(x)$$

Prove by mathematical induction that

$$F_n(x) = \frac{x}{2^n - (2^n - 1)x}$$

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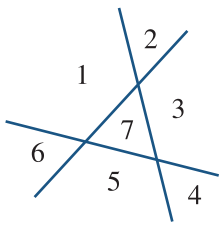
20 Suppose that we draw n lines in a plane so that no three are concurrent and no two are parallel. Let R_n be the number of regions into which these lines divide the plane.

For example, the diagram on the right illustrates that $R_3 = 7$.



a By drawing diagrams, find R_0, R_1, R_2, R_3 and R_4.

b Guess a formula for R_n in terms of n.

c Confirm that your formula is valid by using mathematical induction.

21 In this question, you will prove the **binomial theorem**, which states that

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_{n-1} a^1 b^{n-1} + {}^nC_n a^0 b^n$$

for all $n \in \mathbb{N}$.

a **Pascal's rule** is the identity

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad (\text{where } 1 \le r < n)$$

Prove this identity by using the formula ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$.

b Prove the binomial theorem by using mathematical induction and Pascal's rule.

c Using the binomial theorem, show that $2^n = {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n$ for all $n \in \mathbb{N}$.

22 In Question 8, you proved that if n is an integer, then $n^2 - n$ is even. In this question, you will prove a generalisation of this result. The proof will use the binomial theorem.

a Let p be a prime number and let i be a positive integer less than p. Explain why ${}^pC_i$ is divisible by p.

b **Fermat's little theorem** states:

If p is a prime number and n is any integer, then $n^p - n$ is divisible by p.

Prove this theorem.

Hint: First prove the theorem in the case that n is positive. You can do this by using mathematical induction, the binomial theorem and part **a**.

23 Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} + 3$, where $t_1 = 3$. Prove by induction that $t_n = 3 \times 2^n - 3$.

24 Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} - n$, where $t_1 = 1$. Prove by induction that $t_n = n - 2^n + 2$.

25 Consider a cricket tournament with n teams, where each team plays every other team exactly once. Assume that there are no draws. Show that it is always possible to label the teams $T_1, T_2, \ldots, T_n$ in such a way that

$$T_1 > T_2 > \cdots > T_n$$

where the notation $T_i > T_{i+1}$ means that team T_i beat team T_{i+1}.

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Chapter summary

Assign-ment

Nrich

Basic concepts of proof

- A **conditional statement** has the form: If P is true, then Q is true. This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not Q) $\Rightarrow$ (not P).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A **counterexample** can be used to demonstrate that a universal statement is false.

Proof by mathematical induction

- **Mathematical induction** is used to prove that a statement is true for all natural numbers.
- The basic outline of a proof by mathematical induction is:

  0 Define the proposition $P(n)$ for $n \in \mathbb{N}$.

  1 Show that $P(1)$ is true.

  2 Assume that $P(k)$ is true for some $k \in \mathbb{N}$.

  3 Show that $P(k+1)$ is true.

  4 Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Technology-free questions

1 A **Pythagorean triple** (a, b, c) consists of natural numbers a, b, c such that $a^2 + b^2 = c^2$.

a Let a and d be natural numbers and assume that $(a, a+d, a+2d)$ is a Pythagorean triple. Prove that $a = 3d$.

b Assume that (p, q, r) is a Pythagorean triple, where p is a prime number. Prove that $p = \sqrt{2q+1}$.

2 When you reverse the digits of the three-digit number 435 you obtain 534. The difference between the two numbers is divisible by 9, since

$$534 - 435 = 99 = 9 \cdot 11$$

Prove that this works for any three-digit number.

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3 **a** Find real numbers a and b for which

$$\frac{1}{x(x+3)} = \frac{a}{x} + \frac{b}{x+3}$$

b Hence, evaluate

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{97 \cdot 100}$$

4 Let a and b be positive real numbers.

a Prove that $a > b$ if and only if $a^2 > b^2$.

b Hence, prove that $\sqrt{15} > \sqrt{2} + \sqrt{6}$.

c Also prove that $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$.

5 Let n be an integer with $n \geq 2$. Prove that $\log_n(n+1)$ is irrational.

6 Let n be an integer, and consider the statement:

If $n+1$ is divisible by 3, then n^3+1 is divisible by 3.

a Prove that the statement is true.

b Write down the contrapositive of the statement.

c Write down the converse of the statement.

d Is the converse statement true or false? If it is true, give a proof. Otherwise, give a counterexample.

7 **a** Let b be a non-zero rational number. Prove by contradiction that $\sqrt{2} \cdot b$ is irrational.

Note: You may assume that $\sqrt{2}$ is irrational.

b Hence, prove that every non-zero rational number b can be written as the product of two irrational numbers.

8 Provide a counterexample for each of the following statements:

a If p is an odd prime, then $p+2$ is also an odd prime.

b For all $n \in \mathbb{N}$, if n^3 is divisible by 8, then n is divisible by 8.

c For all real numbers a, b, c and d, if $a < b$ and $c < d$, then $ac < bd$.

d For all $n \in \mathbb{N}$, the numbers n, $n+4$ and $n+6$ cannot all be prime.

9 Let a, b and c be positive integers, and consider the statement:

If $a^2 + b^2 = c^2$, then at least one of a, b or c is even.

a Write down the contrapositive of the statement.

b Prove the contrapositive statement.

10 **a** Prove that the square of any integer n is of the form $3k$ or $3k+1$, where $k \in \mathbb{Z}$.

b Explain why this means that there are no square numbers in the sequence

$$11,\ 101,\ 1001,\ 10001,\ 100001,\ \ldots$$

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11 Prove by mathematical induction that:

a $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$ for all $n \in \mathbb{N}$

b $3^{-1} + 3^{-2} + \cdots + 3^{-n} = \dfrac{3^n - 1}{2(3^n)}$ for all $n \in \mathbb{N}$

12 For every natural number $n \geq 2$, prove that

$$\sum_{j=2}^{n} \frac{4}{j^2 - 1} = \frac{(n-1)(3n+2)}{n(n+1)}$$

13 **a** Prove by induction that $n^3 > 2n + 1$ for all $n \geq 2$.

b Hence, prove by induction that $n! > n^2$ for all $n \geq 4$.

14 Prove by induction that $3^n > n^2 + n$ for all $n \in \mathbb{N}$.

15 Prove each of the following divisibility results for every natural number n:

a $7^{2n-1} + 5$ is divisible by 12

b $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

Multiple-choice questions

1 Suppose that m is odd and n is even. Which one of the following statements is true?

A $m + n$ is even
B $m^2 + n^2$ is even
C $(m + n)^2$ is even
D $3m + 2n$ is even
E $2m + 3n$ is even

2 Let n be an integer, and consider the statement: If n is odd, then n^2 is odd.
The converse of this statement is

A If n is even, then n^2 is odd.
B If n is odd, then n^2 is even.
C If n^2 is odd, then n is even.
D If n^2 is even, then n is odd.
E If n^2 is odd, then n is odd.

3 Let a be an integer, and consider the statement: If $1 + a + a^2$ is odd, then a is even.
The contrapositive of this statement is

A If a is even, then $1 + a + a^2$ is odd.
B If a is odd, then $1 + a + a^2$ is even.
C If $1 + a + a^2$ is even, then a is odd.
D If $1 + a + a^2$ is odd, then a is odd.
E If $1 + a + a^2$ is even, then a is even.

4 Consider the statement: There exists $n \in \mathbb{N}$ such that n is odd and n^2 is even.
The negation of this statement is

A There exists $n \in \mathbb{N}$ such that n is even and n^2 is odd.
B There exists $n \in \mathbb{N}$ such that n is even or n^2 is odd.
C For all $n \in \mathbb{N}$, we have that n is odd and n^2 is even.
D For all $n \in \mathbb{N}$, we have that n is odd or n^2 is even.
E For all $n \in \mathbb{N}$, we have that n is even or n^2 is odd.

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5 Let a, b and c be real numbers such that $ca = cb$. Which one of the following statements *must* be true?

A $a = b$ or $c = 0$ **B** $a = b$ and $c = 1$ **C** $a = b$
D $c = 1$ **E** $c = 0$

6 Consider the statement:

- For every function $f : \mathbb{R} \to \mathbb{R}$, if f is strictly increasing, then the range of f is $\mathbb{R}$.

Which one of the following functions shows that this statement is false?

A $f(x) = 0$ **B** $f(x) = x$ **C** $f(x) = 2^x$ **D** $f(x) = x^2$ **E** $f(x) = x^3$

7 The sum $\sum_{i=3}^{5} i^2$ is equal to

A 12 **B** 24 **C** 48 **D** 50 **E** 60

8 If $\prod_{i=1}^{n} i = 24$, then

A $n = 2$ **B** $n = 3$ **C** $n = 4$ **D** $n = 5$ **E** $n = 6$

Extended-response questions

1 **a** Let $m \in \mathbb{N}$. By expanding the right-hand side, prove that

$$x^m - 1 = (x - 1)(1 + x + x^2 + \cdots + x^{m-1})$$

b Hence, prove the statement:

For all $n \in \mathbb{N}$, if n is not prime, then $2^n - 1$ is not prime.

Hint: If $n \in \mathbb{N} \setminus \{1\}$ and n is not prime, then $n = km$ for some $k, m \in \mathbb{N} \setminus \{1\}$.

c Now consider the converse statement:

For all $n \in \mathbb{N}$, if $2^n - 1$ is not prime, then n is not prime.

Find a counterexample to show that this statement is false.

2 Consider a Pythagorean triple (a, b, c). This means that a, b and c are natural numbers such that $a^2 + b^2 = c^2$, and therefore a, b and c are the side lengths of a right-angled triangle with hypotenuse c.

a Show that if c is odd, then exactly one of a or b is odd.

b Prove that

$$\frac{abc}{a + b + c} = \frac{c(a + b - c)}{2}$$

c Hence, prove that the product of the side lengths of the triangle, abc, is divisible by the perimeter, $a + b + c$.

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3 Consider a function $f : \mathbb{R} \to \mathbb{R}$.

- We say that f is **even** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- We say that f is **odd** if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

For example, the function $f(x) = x^2$ is even, since $f(-x) = (-x)^2 = x^2 = f(x)$.

a Prove that $f(x) = x^3$ is an odd function.

b Prove that the product of two even functions is even.

c Prove that the product of two odd functions is even.

d Prove that the sum of two odd functions is odd.

e Prove that the sum of two even functions is even.

f Prove that the graph of every odd function passes through the origin.

g Find the only function $f : \mathbb{R} \to \mathbb{R}$ that is both even and odd.

4 For $n \in \mathbb{N}$, the nth derivative of $f(x)$ can be written as $f^{(n)}(x)$.

Let $f(x) = \dfrac{1}{2x+1}$. Prove that $f^{(n)}(x) = (-1)^n \dfrac{2^n \cdot n!}{(2x+1)^{n+1}}$.

5 A sequence $a_1, a_2, a_3, \dots$ is given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Using mathematical induction, prove each of the following:

a $a_{n+1} > a_n$ for all $n \in \mathbb{N}$

b $a_n < 2$ for all $n \in \mathbb{N}$

6 Prove that, for any natural number $n \geq 3$, you can find a set A consisting of n natural numbers such that the sum of the numbers in A is divisible by each of the numbers in A.
Hint: This is an induction proof with the base case $n = 3$.

7 We say that a point $P(x, y)$ in the Cartesian plane is a **rational point** if both x and y are rational numbers. The unit circle $x^2 + y^2 = 1$ has infinitely many rational points. For example, the rational points $(1, 0)$ and $(\frac{3}{5}, \frac{4}{5})$ lie on the unit circle.

a Show that the curve $x^2 + y^2 = 3$ has no rational points.
Hint: This is a challenging proof by contradiction.

b Hence, prove that $\sqrt{3}$ is irrational.

c You have shown that the curve $x^2 + y^2 = 3$ has no rational points. Explain why this implies that $x^2 + y^2 = 3^k$ has no rational points, where k is an odd natural number.

d Hence, prove that $\sqrt{3^k}$ is irrational, for every odd natural number k.

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